

The Zero Lower Bound: Frequency, Duration, and Numerical Convergence*

Alexander W. Richter Nathaniel A. Throckmorton

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ABSTRACT

When monetary policy faces a zero lower bound (ZLB) constraint on the nominal interest rate, a minimum state variable (MSV) solution may not exist even if the Taylor principle holds when the ZLB does not bind. This paper shows there is a clear tradeoff between the expected frequency and average duration of ZLB events along the boundary of the convergence region—the region of the parameter space where our policy function iteration algorithm converges to an MSV solution. We show this tradeoff with two alternative stochastic processes: one where monetary policy follows a 2-state Markov chain, which exogenously governs whether the ZLB binds, and the other where ZLB events are endogenous due to discount factor or technology shocks. We also show that small changes in the parameters of the stochastic processes cause meaningful differences in the decision rules and where the ZLB binds in the state space.

Keywords: Monetary policy; zero lower bound; convergence; minimum state variable solution; policy function iteration

JEL Classifications: E31; E42; E58

*Richter, Department of Economics, Auburn University, 0332 Haley Center, Auburn, AL (arichter@auburn.edu); Throckmorton, Department of Economics, Indiana University and DePauw University, 7 E Larabee St, Harrison Hall, Greencastle, IN (nathrock@indiana.edu). We thank Eric Leeper and Todd Walker for helpful comments on an earlier draft. We would also like to thank two anonymous referees for helpful comments on a previous draft.

1 INTRODUCTION

Since the beginning of the Great Recession in late 2008, many central banks around the world have targeted a policy rate near zero and promised to maintain a low rate until economic conditions improve. Despite this policy and numerous unconventional policies, most of these countries face elevated unemployment levels and anemic growth five years later. This experience has ignited new research that studies the impacts of the zero lower bound (ZLB) on the nominal interest rate.

A ZLB constraint is similar to a monetary policy rule that occasionally pegs the nominal interest rate, but where households never expect the interest rate to fall below zero. If the central bank does not switch rules, then it is well known in a linear model that indeterminacy occurs when the Taylor (1993) principle (i.e., the principle that monetary policy pins down prices by adjusting the nominal interest rate more than one-for-one with inflation) does not hold. This means that if the household expects the central bank to always peg the nominal interest rate, then the price level, and hence inflation, is not pinned down. If the household expects the central bank to occasionally peg the nominal interest rate, then the fraction of time the central bank satisfies the Taylor principle *may* provide enough price stability to deliver a determinate equilibrium [Davig and Leeper (2007)].

We first show the convergence region—the region of the parameter space where our policy function iteration algorithm converges to a minimum state variable (MSV) solution—is identical to the determinacy region that Davig and Leeper (2007) derive in a Fisherian economy with a Markov-switching monetary policy rule. We then locate the convergence region in a nonlinear New Keynesian model with a ZLB constraint. The boundary of the convergence region imposes a clear tradeoff between the expected frequency and average duration of ZLB events. We show this tradeoff with two alternative stochastic processes: one where monetary policy follows a 2-state Markov chain, which exogenously governs whether the ZLB binds, and the other where ZLB events are endogenous due to technology or discount factor shocks. We also show that small changes in the parameters of the stochastic processes cause meaningful differences in the decision rules and where the ZLB binds in the state space, which affect estimation and policy analysis.

Within the class of linear Markov-switching rational expectations models, Farmer et al. (2009, 2010), Barthélemy and Marx (2013), and Cho (2013) prove that non-MSV solutions may exist even when the MSV solution is determinate. To be clear, our algorithm does not converge in regions of the parameter space that are typically considered indeterminate (e.g., in fixed regime models without a ZLB constraint, our algorithm only converges when the Taylor principle is satisfied), but it cannot capture any non-MSV solutions that may exist when a convergent MSV solution exists.¹ Studying non-MSV solutions in models with a ZLB constraint is an important research topic, but we believe locating regions of the parameter space that deliver a convergent MSV solution is significant since most macroeconomic research, including estimation, is based on MSV solutions.

The ZLB constraint imposes an unavoidable nonlinearity in the monetary policy rule. The literature has relied on several different techniques to deal with this challenge. One common technique is to break the problem into pre- and post-ZLB periods [e.g., Braun and Körber (2011); Braun and Waki (2006); Christiano et al. (2011); Eggertsson and Woodford (2003); Erceg and Linde (2010); Gertler and Karadi (2011)]. With this approach, a large unanticipated shock causes the ZLB to bind. Each period, there is a probability that the nominal interest rate exits the ZLB. Once the nominal interest rate exits the ZLB, there is no chance of returning. The drawback with this simplifying assumption is that if a shock causes the ZLB to bind in one period, there is no

¹Barthélemy and Marx (2013) refer to unique bounded MSV solutions as bounded Markovian solutions.

reason to expect that the same shock would not cause the ZLB to bind in a future period. Much of the literature also linearizes the equilibrium system, except the monetary policy rule, which causes approximation error [Braun et al. (2012); Fernández-Villaverde et al. (2012); Gavin et al. (2014)]. These approaches make the algorithm numerically tractable because they do not rely on a grid-based solution method, but they also have drawbacks which motivate solving the fully nonlinear model to accurately account for the expectational effects of going to and leaving the ZLB.

A recent segment of the ZLB literature uses global solution methods to solve fully nonlinear models with a ZLB constraint [e.g., Aruoba and Schorfheide (2013); Basu and Bundick (2012); Fernández-Villaverde et al. (2012); Gavin et al. (2014); Gust et al. (2013); Mertens and Ravn (2013); Nakata (2012); Richter et al. (2013); Wolman (2005)]. However, all of the work on determinacy uses a perfect foresight setup [e.g., Alstadheim and Henderson (2006); Benhabib et al. (2001a,b)]. Although we do not provide formal proofs, our numerical convergence regions demonstrate the restrictions these nonlinear solution methods face and the challenges of estimating constrained nonlinear models with a particle filter [Fernández-Villaverde and Rubio-Ramírez (2007)].

The paper is organized as follows. [Section 2](#) provides some numerical and analytical evidence for the link between determinacy and convergence in a simple Fisherian economy. [Section 3](#) lays out the constrained nonlinear model, baseline calibration, and our solution procedure. [Sections 4](#) and [5](#) define the two alternative stochastic processes that drive the economy to the ZLB and show the tradeoff between the frequency and average duration of ZLB events. [Section 6](#) concludes.

2 THE LINK BETWEEN DETERMINACY AND CONVERGENCE: SOME EVIDENCE

Davig and Leeper (2007) study determinacy in linear models that do not include a ZLB constraint. Their models contain two monetary policy rules—one that aggressively responds to inflation and one that reacts less aggressively to inflation—governed by a 2-state Markov chain. The special case where the central bank pegs the nominal interest rate in one regime and obeys the Taylor principle in the other regime is similar to a model with a ZLB constraint. Thus, we use their regime switching setup as a benchmark for our algorithm. When we adopt their models (linear Fisherian economy, linear New Keynesian economy), our algorithm produces convergence regions that are identical to the determinacy regions they analytically derive. This means our algorithm is non-convergent whenever the monetary policy parameters are outside their analytical determinacy region and convergent whenever the Long-run Taylor Principle is met. Our numerical solutions to these models also equal the MSV solutions they derive. This exercise does not constitute a formal proof, but it does provide evidence that our algorithm captures determinate MSV solutions.

Our finding that there exists a tradeoff between the expected frequency and average duration of ZLB events is similar to the conclusion in Davig and Leeper (2007). They prove that when there are distinct monetary policy regimes, the Taylor principle does not need to hold in both regimes to guarantee a unique bounded MSV solution. As long as one of the regimes satisfies the Taylor principle, the central bank can passively respond to inflation (i.e., adjust the nominal interest rate less than one-for-one with inflation) in the other regime and still deliver a determinate solution. However, there are two key differences between our setups. First, an occasionally binding ZLB constraint truncates the current and future nominal interest rate distributions, which affects the household's expectations and their decision rules. Second, the parameters of the exogenous driving processes affect convergence, since the linearized version of a nonlinear model with a ZLB constraint misses key interaction terms between exogenous variables and expected inflation.

To see how the parameters of the exogenous driving process matter for convergence, consider the nonlinear analogue of the Fisherian economy Davig and Leeper (2007) study. A representative household chooses $\{c_t, b_t\}_{t=0}^{\infty}$ to maximize $E_0 \sum_{t=0}^{\infty} \beta_t \log c_t$, where c_t is consumption, $\tilde{\beta}_0 \equiv 1$, and $\tilde{\beta}_t = \prod_{i=1}^t \beta_i$ for $t > 0$. These choices are constrained by $c_t + b_t = y + i_{t-1} b_{t-1} / p_t$, where y is a constant endowment, b_t is a one-period nominal bond, i_{t-1} is the gross nominal interest rate set by the central bank, and $\pi_t = p_t / p_{t-1}$ is the gross inflation rate. The fiscal authority does not issue debt so bonds are in zero-net supply. The equilibrium system is composed of

$$1 = i_t E_t[\beta_{t+1} / \pi_{t+1}], \quad (1)$$

$$i_t = \bar{i} (\pi_t / \bar{\pi})^{\phi(s_t)}, \quad (2)$$

$$\beta_t = \bar{\beta} (\beta_{t-1} / \bar{\beta})^{\rho_\beta} \exp(v_t), \quad (3)$$

where β_t is the discount factor, which evolves according to (3) with $|\rho_\beta| < 1$ and $v_t \sim \mathbb{N}(0, \sigma_v^2)$. $\phi(s_t)$ is the policy response to changes in inflation, which follows a 2-state Markov chain with transition matrix $\Pr\{s_t = j | s_{t-1} = i\} = p_{ij}$, $i, j \in \{1, 2\}$. A bar denotes a steady-state value.

A second-order approximation of (1) around the deterministic steady state implies

$$\hat{i}_t + \underbrace{(\hat{i}_t - E_t[\hat{\pi}_{t+1}] + E_t[\hat{\beta}_{t+1}])^2}_{=0 \text{ (First Order)}} = E_t[\hat{\pi}_{t+1}] - E_t[\hat{\beta}_{t+1}] - \underbrace{(E_t[(\hat{\pi}_{t+1} - \hat{\beta}_{t+1})^2] - (E_t[\hat{\pi}_{t+1} - \hat{\beta}_{t+1}])^2)}_{=0 \text{ (First Order, Jensen's Inequality)}}, \quad (4)$$

where a hat denotes log deviation from the steady-state value. Up to a first order, this equation reduces to the standard log-linear Fisher equation, which, when combined with (2), reduces to

$$\phi(s_t) \hat{\pi}_t = E_t[\hat{\pi}_{t+1}] - E_t[\hat{\beta}_{t+1}].$$

If the monetary policy regime is fixed ($\phi(s_t) = \phi$), determinacy requires $\phi > 1$ (Taylor principle). If monetary policy is state-dependent ($\phi(s_t = i) = \phi_i$), determinacy in the linear model requires

$$p_{11}(1 - \phi_2) + p_{22}(1 - \phi_1) + \phi_1 \phi_2 > 1. \quad (\text{Long-run Taylor Principle})$$

Neither of these conditions include the parameters of the discount factor process. This is a byproduct of first-order approximations, which remove all interaction terms between the expected discount factor and expected inflation. With a higher order approximation, such as the second-order approximation in (4), these interaction terms appear and affect convergence. When fluctuations in the discount factor are more persistent, it causes more persistent deviations of inflation from its steady state, which shrinks the convergence region. As an example, [figure 1](#) shows the convergence regions (shaded) for the state-dependent log-linear model and the nonlinear model with $\rho_\beta = 0.85$ and $\rho_\beta = 0.95$ in (ϕ_1, ϕ_2) -space.² The convergence region is smaller in the nonlinear model and decreases with ρ_β . However, changes to σ_β do not influence the convergence region since it only affects the magnitude of the shock and not the household's consumption/saving decision.

In models with a ZLB constraint, both the persistence and standard deviation of the exogenous driving processes affect the convergence region. It is well known that these models contain two deterministic equilibria [Benhabib et al. (2001a,b)]. Specifically, there are two nominal interest rate/inflation rate pairs consistent with the steady-state equilibrium system. In one case the central bank meets its positive inflation target, while in the other, deflation occurs. Similar to the sunspot

²For the purposes of this exercise, we fix $\bar{\beta} = 0.99$, $\bar{\pi} = 1.005$, $p_{11} = 0.8$, $p_{22} = 0.95$, and $\sigma_v = 0.0005$.

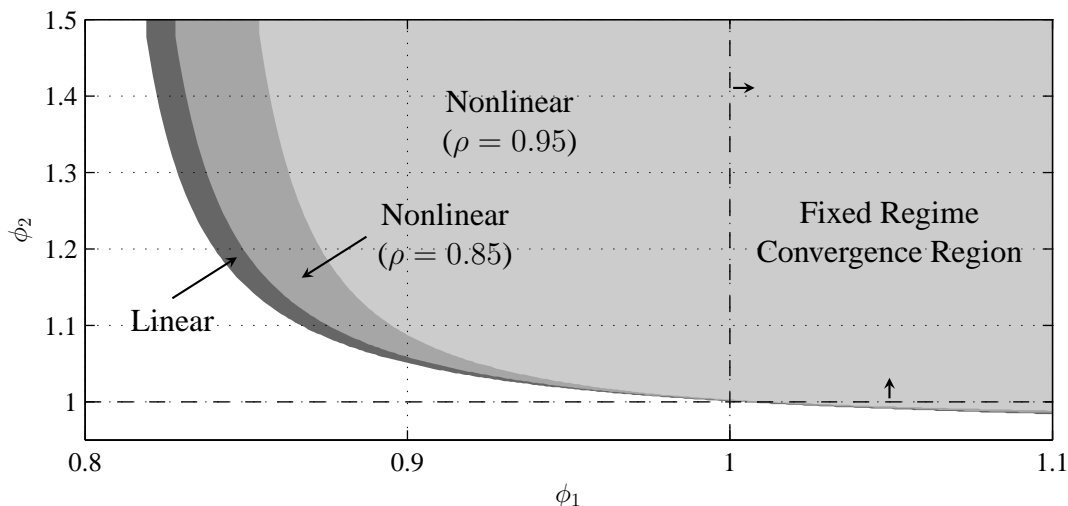


Figure 1: Fisherian economy convergence (shaded) regions in (ϕ_1, ϕ_2) -space. We set $(p_{11}, p_{22}) = (0.8, 0.95)$.

shocks in Aruoba and Schorfheide (2013) and the confidence shocks in Mertens and Ravn (2013), exogenous switches in the monetary policy state that occur in our model cause the economy to switch between two states, but it does *not* necessarily imply that multiple MSV solutions exist in a stochastic economy.³ As long as there is a sufficient expectation of returning to a monetary policy rule that obeys the Taylor principle, we show that an MSV solution exists. Our algorithm does not converge to the deflationary equilibrium because it is not stable in expectation. That is, in the deflationary equilibrium there is not enough price stability to reestablish that equilibrium when sufficiently large shocks hit the economy, which is similar to points that Christiano and Eichenbaum (2012) make about E-learnability. They contend that when the model with a ZLB constraint is restricted to the class of equilibria that are E-learnable, it has a unique solution.

To understand this point more clearly, suppose the economy begins in a state where the nominal interest rate is stuck at its ZLB and inflation is negative. In a deterministic environment, households have dogmatic expectations, meaning that they never expect to leave this state, and an equilibrium exists. This fact is in sharp contrast with a stochastic environment, where households form expectations over the complete distribution of shocks. In our exogenous and endogenous ZLB setups, households place probability mass on both the nominal interest rate being positive and zero in expectation, so that on average they always expect a positive nominal interest rate. Thus, the deflationary state cannot be an equilibrium since the expected nominal interest rate is never zero, and the stochastic economy will always gravitate toward the positive inflation equilibrium.

3 MODEL, BASELINE CALIBRATION, AND SOLUTION METHOD

At the ZLB, monetary policy cannot directly affect the real interest rate to stabilize inflation. Without price adjustment costs or sticky prices, such as in the Fisherian economy described above, no region of the parameter space delivers a convergent solution even if ZLB events are infrequent. In a new Keynesian model, nominal frictions anchor prices at the ZLB so that a strong enough expectation of leaving the ZLB produces a convergent solution. We show the convergence regions under

³Aruoba and Schorfheide (2013) discuss sunspot equilibria, but these are *not* the same sunspots Farmer et al. (2009, 2010) and Cho (2013) emphasize, since they omit the non-MSV component from their solution.

alternative parameterizations using a conventional New Keynesian model with a ZLB constraint.

3.1 MODEL A representative household chooses $\{c_t, n_t, b_t\}_{t=0}^{\infty}$ to maximize expected lifetime utility, given by, $E_0 \sum_{t=0}^{\infty} \tilde{\beta}_t \{c_t^{1-\sigma}/(1-\sigma) - \chi n_t^{1+\eta}/(1+\eta)\}$, where $1/\sigma$ is the elasticity of intertemporal substitution, $1/\eta$ is the Frisch elasticity of labor supply, c_t is consumption of the final good, n_t is labor hours, $\tilde{\beta}_0 \equiv 1$, and $\tilde{\beta}_t = \prod_{i=1}^t \beta_i$ for $t > 0$. β_i is the subjective discount factor in period i . These choices are constrained by $c_t + b_t + \tau_t = w_t n_t + r_{t-1} b_{t-1}/\pi_t + d_t$, where $\pi_t = p_t/p_{t-1}$ is the gross inflation rate, w_t is the real wage rate, τ_t is a lump-sum tax, b_t is a one-period real bond, r_t is the gross nominal interest rate, and d_t are profits from intermediate firms. The optimality conditions to the household's problem imply

$$w_t = \chi n_t^\eta c_t^\sigma, \quad (5)$$

$$1 = r_t E_t \{ \beta_{t+1} (c_t/c_{t+1})^\sigma / \pi_{t+1} \}. \quad (6)$$

The production sector consists of a continuum of monopolistically competitive intermediate goods firms and a perfectly competitive final goods firm. Each firm $i \in [0, 1]$ in the intermediate goods sector produces a differentiated good, $y_t(i)$, according to $y_t(i) = a_t n_t(i)$, where a_t is technology and $n_t(i)$ is the level of employment used by firm i . The final goods firm purchases $y_t(i)$ units from each intermediate firm to produce the final good, $y_t \equiv [\int_0^1 y_t(i)^{(\theta-1)/\theta} di]^\theta / (\theta-1)$, according to a Dixit and Stiglitz (1977) aggregator, where $\theta > 1$ is the price elasticity of demand between intermediate goods. Profit maximization yields the demand function for good i , $y_t(i) = (p_t(i)/p_t)^{-\theta} y_t$, where $p_t = [\int_0^1 p_t(i)^{1-\theta} di]^{1/(1-\theta)}$ is the final good price. Each intermediate firm chooses its price level, $p_t(i)$, to maximize the expected present value of real profits, $E_t \sum_{k=t}^{\infty} q_{t,k} d_k(i)$, where $q_{t,t} \equiv 1$, $q_{t,t+1} = \beta_{t+1} (c_t/c_{t+1})^\sigma$ is the pricing kernel between periods t and $t+1$, and $q_{t,k} \equiv \prod_{j=t+1}^k q_{j-1,j}$. Following Rotemberg (1982), each firm faces a cost to adjusting its price, which emphasizes the potentially negative effect that price changes can have on customer-firm relationships. Using the functional form in Ireland (1997), firm i 's real profits are

$$d_t(i) = \left[\left(\frac{p_t(i)}{p_t} \right)^{1-\theta} - \frac{w_t}{a_t} \left(\frac{p_t(i)}{p_t} \right)^{-\theta} - \frac{\varphi}{2} \left(\frac{p_t(i)}{\bar{\pi} p_{t-1}(i)} - 1 \right)^2 \right] y_t,$$

where $\varphi \geq 0$ determines the size of the adjustment cost, w_t/a_t is the real marginal cost of producing a unit of output, and $\bar{\pi}$ is the steady-state gross inflation rate. In a symmetric equilibrium, all intermediate goods firms make the same decisions and the optimality condition reduces to

$$\varphi \left(\frac{\pi_t}{\bar{\pi}} - 1 \right) \frac{\pi_t}{\bar{\pi}} = (1-\theta) + \theta(w_t/a_t) + \varphi E_t \left[q_{t,t+1} \left(\frac{\pi_{t+1}}{\bar{\pi}} - 1 \right) \frac{\pi_{t+1}}{\bar{\pi}} \frac{y_{t+1}}{y_t} \right]. \quad (7)$$

In the absence of price adjustment costs (i.e., $\varphi = 0$), the real marginal cost equals $(\theta-1)/\theta$, which is the inverse of the firm's markup of price over marginal cost.

Each period the fiscal authority finances its spending, \bar{g} , by levying lump-sum taxes ($\tau_t = \bar{g}$). The resource constraint is $c_t + \bar{g} = [1 - \varphi(\pi_t/\bar{\pi} - 1)^2/2] y_t \equiv y_t^{adj}$, where y_t^{adj} includes the value added by intermediate firms, which is their output minus quadratic price adjustment costs. A competitive equilibrium consists of sequences of quantities $\{c_t, n_t, b_t, y_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t, \pi_t\}_{t=0}^{\infty}$, and exogenous variables $\{\beta_t, a_t\}_{t=0}^{\infty}$ that satisfy the household's and firm's optimality conditions [(5),(6),(7)], the production function, $y_t = a_t n_t$, the monetary policy rule (defined below), the stochastic processes (defined below), bond market clearing, $b_t = 0$, and the resource constraint.

3.2 CALIBRATION We calibrate the model at a quarterly frequency using values that are common in the literature. We set $\bar{\beta} = 0.99$ and $\sigma = 1$, implying log utility in consumption. The Frisch elasticity of labor supply, $1/\eta$, is set to 1 and the leisure preference parameter, χ , is set so that steady-state labor equals $1/3$ of the available time. The price elasticity of demand between intermediate goods, θ , is set to 6, which corresponds to an average markup of price over marginal cost equal to 20 percent. The costly price adjustment parameter, φ , is set to 58.25, which is similar to a Calvo (1983) price-setting specification where prices change on average once every four quarters ($\omega = 0.75$).⁴ Steady-state technology, \bar{a} , is normalized to 1. In the policy sector, the steady-state gross inflation rate, $\bar{\pi}$, is set to 1.005, which implies an annual (net) inflation rate target of 2 percent. The steady-state ratio of government spending to output is calibrated to 20 percent.

3.3 SOLUTION METHOD AND DEFINITION OF CONVERGENCE We solve the fully nonlinear model using the policy function iteration algorithm described in Richter et al. (2013), which is a numerical byproduct of using monotone operators to prove existence and uniqueness of equilibria.⁵ This solution method discretizes the state space and uses time iteration to solve for the updated decision rules until the tolerance criterion is met. To account for the ZLB in the endogenous setup, we set the gross nominal interest rate equal to 1 on any node in the state space where the Taylor rule implies a value less than one. We obtain initial conjectures for the constrained nonlinear model using the solution to the log-linear model without the ZLB imposed. We find that this guess is very reliable and no evidence it affects convergence. For example, when we solve for the boundary of the convergence region using this guess for each parameterization, it produces the same boundary as when we use the nonlinear solution to a model with a similar parameterization as our guess.

We classify the algorithm as *non-convergent* whenever the iteration step, defined as the maximum distance between decision rule values on successive iterations, increases at an increasing rate for more than 100 iterations or when all of the values in any decision rule consistently drift (e.g., negative consumption on any node or more than 50 percent deflation on every node). Additionally, when ZLB events are endogenous, we require that the ZLB binds on fewer than 50 percent of the nodes in the state space. We have observed that the percentage of nodes where the ZLB binds converges to 100 percent, whenever more than 50 percent of the nodes bind in an iteration. At that point, the inflation policy function converges to zero at all nodes in the state space, which is not a valid solution. We classify the algorithm as *convergent* whenever the iteration step is less than 1^{-13} (the tolerance criterion) for 10 successive iterations, which prevents the algorithm from immediately converging when the tolerance criterion is first met. To provide evidence that each MSV solution is locally unique, we randomly perturb the converged decision rules in multiple directions and check that the algorithm converges back to the same solution.⁶ To ensure that the solution is

⁴If ω represents the fraction of firms that cannot adjust prices each period, then $\varphi = \omega(\theta - 1)/[(1 - \omega)(1 - \beta\omega)]$ in a linear model with a zero-inflation steady state, which provides a reasonable estimate of the adjustment cost parameter.

⁵Coleman (1991) proves existence and uniqueness of an equilibrium in a nonlinear stochastic production economy with an income tax. Greenwood and Huffman (1995) adapt this proof to a more general neoclassical model, including one with monopolistic competition. Coleman (1997) generalizes these proofs to allow for an endogenous labor supply and Datta et al. (2005, 2002); Mirman et al. (2008) extends them to more complex setups. The monotone mapping results in these papers are attractive because they serve as the theoretical foundations of our numerical algorithm.

⁶Cochrane (2011) argues that the existence of explosive inflation paths in a New Keynesian model permit a bounded solution. Our numerical solution method cannot capture explosive paths, which are not observed in the data. We focus on conventional bounded equilibria. Braun et al. (2012) demonstrate that there are multiple equilibria for *some* settings of parameters and shocks. For example, a second equilibrium exists if a shock to the discount factor is greater than

bounded, we simulate the model and check that it converges to a stochastic steady state. For a more formal description of the numerical algorithm and convergence, see appendix A.

4 EXOGENOUS ZLB EVENTS: MONETARY POLICY SWITCHING

In this section, the central bank sets the gross nominal interest rate according to

$$r_t = \begin{cases} \bar{r}(\pi_t/\bar{\pi})^{\phi_\pi} (y_t^{adj}/\bar{y})^{\phi_y} & \text{for } s_t = 1 \\ 1 & \text{for } s_t = 2 \end{cases}, \quad (8)$$

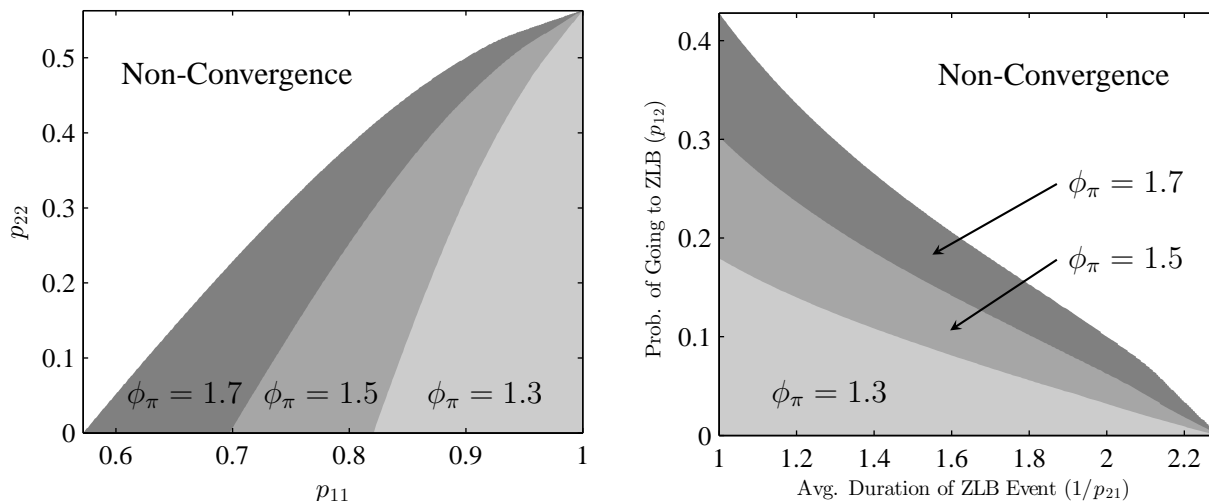
where ϕ_π and ϕ_y are the policy responses to inflation and the adjusted output gap. The monetary policy state, s_t , evolves according to a 2-state Markov chain with transition matrix $\Pr\{s_t = j | s_{t-1} = i\} = p_{ij}$, for $i, j \in \{1, 2\}$. When $s_t = 1$, the central bank obeys the Taylor principle and when $s_t = 2$, the central bank exogenously pegs the gross nominal interest rate at 1. We set $a_t = \bar{a}$ and $\beta_t = \bar{\beta}$. Thus, all ZLB events in this section are due to exogenous changes in s_t .

The exogenous switches between the two monetary policy states are similar to large discretionary shocks. When the nominal interest rate switches from state 1 to state 2 (state 2 to state 1), the nominal interest rate falls (rises) sharply. This means that expectations about the future state play a key role in determining inflation. To understand how inflation changes when ZLB events are exogenous, assume the state is fixed, but there is a probability it changes. When $s_t = 1$ and $p_{11} < 1$, the household expects a lower future nominal interest rate, which increases expected future consumption growth and drives up inflation. When $s_t = 2$, the household expects to leave the ZLB and the future nominal interest rate to rise. This reduces expected future consumption growth, which would normally reduce inflation, but since the nominal rate is stuck at 1, inflation rises to clear the bond market. Thus, the possibility of ZLB events increases inflation in both states.

Figure 2a plots the convergence (shaded) regions in (p_{11}, p_{22}) -space for $\phi_\pi \in \{1.3, 1.5, 1.7\}$. To isolate the impact of ϕ_π on the convergence region, we initially set $\phi_y = 0$. The boundary of the shaded region for each ϕ_π represents the largest p_{22} value that yields a convergent solution for each p_{11} value. These results show a clear tradeoff between p_{11} and p_{22} . When there is a low probability of going to the ZLB (i.e., a high p_{11} value), it is possible to have a high probability of staying at the ZLB (i.e., a high p_{22} value) and still guarantee a convergent solution. This suggests that there is a tradeoff between the expected frequency and average duration of ZLB events. To see this more clearly, figure 2b plots the probability of going to the ZLB (i.e., p_{12}) as a function of the average duration of each ZLB event (i.e., $1/p_{21}$) for each value of ϕ_π . When the average duration of ZLB events is short, the convergence region permits a high expected frequency of ZLB events. However, as the average duration of ZLB events increases, the maximum expected frequency of ZLB events must decrease to avoid the non-convergence (non-shaded) region of the parameter space.

These results show that this model does not generate *average* ZLB events that are consistent with observed ZLB events, which is similar to the points made in Chung et al. (2012) and Fernández-Villaverde et al. (2012); however, it is possible for longer ZLB events to occur and still deliver a convergent solution, because the household places little weight on these outcomes in their expectations. For example, when $\phi_\pi = 1.5$, $p_{11} = 0.95$, and $p_{22} = 0.5$, the average ZLB event is only 2 quarters, but the maximum ZLB event in a 500,000 quarter simulation is 15 quarters, which

7.31 percent. Given the parameters we adopt, a change in the discount factor of even 1.5 percent is very unlikely.


 (a) Convergence (shaded) regions in (p_{11}, p_{22}) -space

(b) Expected frequency/average duration of ZLB events tradeoff

 Figure 2: Convergence regions across alternative monetary policy responses to inflation (ϕ_π).

is closer to ZLB events observed in the data. Furthermore, Gust et al. (2013) argue that this average duration is consistent with the expectations found in financial market and survey data.

The convergence region also depends on how strongly the central bank responds to inflation when the ZLB does not bind. The darker shaded regions represent the additional area of the parameter space that delivers a convergent solution when ϕ_π increases. If the monetary authority responds more aggressively to inflation when $s_t = 1$ (i.e., a higher ϕ_π) and $p_{11} < 1$, the convergence region widens, since greater price stability when $s_t = 1$ helps offset the destabilizing influence of $s_t = 2$. This means the convergence region permits longer and/or more frequent trips to the ZLB. However, it is interesting that regardless of the value of ϕ_π , the longest average ZLB event inside the convergence region is the same (2.3 quarters). As p_{11} rises, the expected frequency of ZLB events declines. This implies that $s_t = 2$ has a decreasing effect on $s_t = 1$ and the stabilizing effect of additional price stability in $s_t = 1$ has a smaller effect on overall price stability. Thus, the additional area of the parameter space that delivers convergence shrinks as p_{11} increases. When $p_{11} = 1$, any ZLB event is completely unexpected by the household. This means that $s_t = 2$ has no effect on the decision rules in $s_t = 1$ and increases in ϕ_π do not widen the convergence region. In short, as $p_{11} \rightarrow 1$, the model approaches a fixed-regime setup where increases in ϕ_π beyond a minimum threshold have no effect on the convergence region in the parameter space.

When the central bank responds to the adjusted output gap ($\phi_y > 0$), it also affects the convergence region. Figure 3 plots these regions for $\phi_y \in \{0, 0.1, 0.2\}$. Since the real interest rate is higher in state 1, adjusted output is below its steady state, which corresponds to a negative output gap (i.e., $y_t^{adj}/\bar{y} < 0$). Thus, a larger ϕ_y reduces the nominal interest rate in state 1 for a given ϕ_π , because the countercyclical monetary policy offsets part of the response to changes in inflation. With less price stability in state 1, the convergence region shrinks. Thus, higher values of ϕ_y have a qualitatively similar effect on the convergence region as lower values of ϕ_π when ZLB events are exogenous. Once again, the differences between these convergence regions shrink as p_{11} increases.

In general, a larger response by the central bank to increases in household demand provides additional price stability in state 1. However, it is destabilizing in state 2, because the household

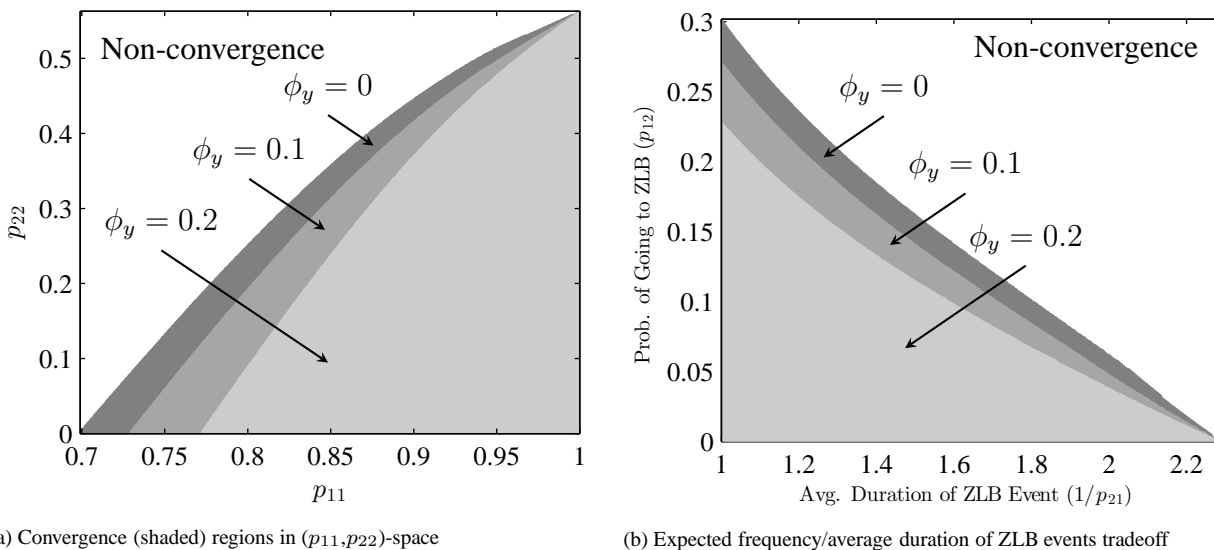


Figure 3: Convergence regions across alternative monetary policy responses to the adjusted output gap (ϕ_y).

expects a relatively higher future nominal interest rate, which increases inflation at the ZLB. The strengths of these two competing effects change along the edge of the convergence region. When p_{11} is not too high, state 2 has a larger effect on state 1. This means the maximum value of p_{22} and the average duration of ZLB events is low. Thus, the stabilizing effect of state 1 dominates and the convergence region expands. As p_{11} increases, the effect of state 2 on state 1 declines and the convergence region expands less. At high values of p_{11} , which permit high values of p_{22} , it is possible that the destabilizing effect of state 2 dominates and for the convergence region to shrink. This is what happens with large values of ϕ_π .⁷ Regardless, when $p_{11} = 1$, state 2 has no effect on state 1 and the monetary policy parameters have no impact on the convergence region.

A common theme in the convergence regions shown above is that price stability plays a key role. In addition to the monetary policy parameters, the degree of price stickiness also heavily influences price stability. Figure 4 shows the convergence regions for $\omega \in \{0.67, 0.75\}$. With a lower degree of price stickiness (i.e., a lower ω), firms have a greater ability to adjust prices with the monetary policy state. When the average duration of ZLB events is high (i.e., a high p_{22}), lower price stickiness shrinks the convergence region, because expected prices are less anchored by the Taylor rule in state 1. For example, if ω declines from 0.75 to 0.67, the maximum average duration of ZLB events declines from 2.3 quarters to 1.85 quarters. As p_{11} and p_{22} fall, the household expects to visit the ZLB more often but for fewer quarters on average. This means the benefit of additional price stability declines and the convergence regions shrink. Notice that the convergence regions twist. While the convergence region generally shrinks for lower values of ω , at low enough values of p_{11} the region expands. This is because it is less costly for firms to adjust prices consistent with state 2 when the expected duration of staying in state 2 is very short.

The other deep parameters in the model (e.g., σ , η , $\bar{\beta}$) also affect the size of the convergence region. When the degree of risk aversion, σ , is higher, the household is less willing to intertemporally substitute consumption goods. When the Frisch elasticity of labor supply, $1/\eta$, is larger, the

⁷Barthélemy and Marx (2013) find that a strong response to inflation shrinks the determinacy region in a linear model with a Markov-switching monetary policy rule. The intuition for their result is similar to what we describe.

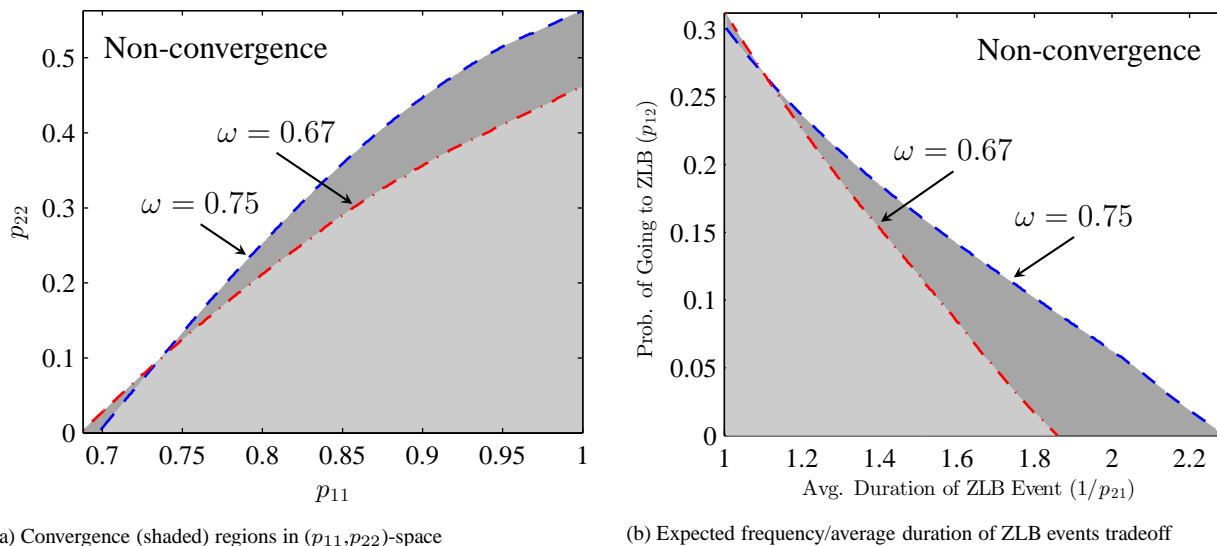


Figure 4: Convergence regions across alternative degrees of price stickiness (ω).

household's willingness to supply labor is more sensitive to changes in the real wage rate. Both of these effects make hours worked, consumption, and the inflation rate less volatile when the ZLB binds, which expands the convergence region. When the household is more patient (i.e., a higher β), the steady state nominal interest rate is lower, which reduces the demand-side effects of switching states. This makes inflation less volatile and also expands the convergence region. Many models also include a smoothing component in the monetary policy rule. An increase in this parameter shrinks the convergence region because it reduces the response to the fundamentals.

5 ENDOGENOUS ZLB EVENTS: EXOGENOUS SHOCKS

This section replaces the exogenous Markov-switching process, given in (8), with either an AR(1) technology or discount factor process that determines the expected frequency and average duration of ZLB events. We study these two shocks because they are the most common shocks used in the ZLB literature.⁸ The central bank sets the gross nominal interest rate according to

$$r_t = \max\{1, \bar{r}(\pi_t/\bar{\pi})^{\phi_\pi}\}. \quad (9)$$

Unlike Eggertsson and Woodford (2003) and others, we chose to use a continuous processes for technology or the discount factor rather than a two-state Markov chain for two main reasons. First, the results are more relevant to researchers interested in estimating ZLB models, since most estimation specifies continuous processes. Second, even with a two-state Markov chain, it would be difficult to compare the results from the exogenous and endogenous setups. For example, with a two-state Markov chain on the discount factor, the jump necessary to make the ZLB bind is a function of the transition matrix. Thus, it is not possible to fix a β in state 2 that ensures the ZLB binds over the entire convergence region in (p_{11}, p_{22}) -space. In other words, the β in state 2 would sometimes be too low to make the ZLB bind and sometimes too high, which affects convergence. Thus, β in state 2 would need to change over the (p_{11}, p_{22}) -space. However, that change make it difficult to directly compare the results to the results from the exogenous setup.

⁸For a complete picture of the solution to New Keynesian models with and without capital see Gavin et al. (2014).

Section 4 makes clear that when episodes at the ZLB are exogenous, the boundary of the convergence region imposes a tradeoff between the expected frequency and average duration of ZLB events. This same tradeoff exists when ZLB events are endogenous. We discretize the state, z_{t-1} (either technology, a_{t-1} , or the discount factor, β_{t-1}), into N elements such that $z_{t-1} \in \{z^1, \dots, z^N\}$. Let $s_t \in \{1, 2\}$ indicate that the ZLB is either not binding or binding, respectively. Given the MSV solution to the model under a particular parameterization, let n denote the index corresponding to the minimum value of the state variable where the ZLB binds, which partitions the state-space into two subsets. Denote the corresponding sets of indices as $\mathcal{I}_{1,t-1} = \{1, \dots, n-1\}$ and $\mathcal{I}_{2,t-1} = \{n, \dots, N\}$. The probability of going to the ZLB (the analog of $p_{12} = 1 - p_{11}$ in the transition matrix defined in section 4) is given by

$$\Pr\{s_t = 2 | s_{t-1} = 1\} = \frac{\sum_{i \in \mathcal{I}_{1,t-1}} \Pr\{s_t = 2 | z_{t-1} = z^i\} \phi(z^i | \bar{z}, \sigma_z)}{\sum_{i \in \mathcal{I}_{1,t-1}} \phi(z^i | \bar{z}, \sigma_z)},$$

where

$$\Pr\{s_t = 2 | z_{t-1} = z^i\} = \frac{\sum_{j \in \mathcal{J}_{2,t}(i)} \phi(\varepsilon_j | 0, \sigma_\varepsilon)}{\sum_{j=1}^M \phi(\varepsilon_j | 0, \sigma_\varepsilon)} = \pi^{-1/2} \sum_{j \in \mathcal{J}_{2,t}(i)} \phi(\varepsilon_j | 0, \sigma_\varepsilon), \quad (10)$$

and $\phi(x | \mu, \sigma)$ is the normal probability density function, given mean μ and standard deviation σ . For each z_{t-1} , there is a vector of realizations of z_t , where each realization corresponds to a Gauss-Hermite quadrature node, ε_j , $j \in \{1, \dots, M\}$ (the roots of the Hermite polynomial). $\mathcal{J}_{2,t}(i)$ is the set of indices where the ZLB continues to bind given the technology state $z_{t-1} = z^i$.

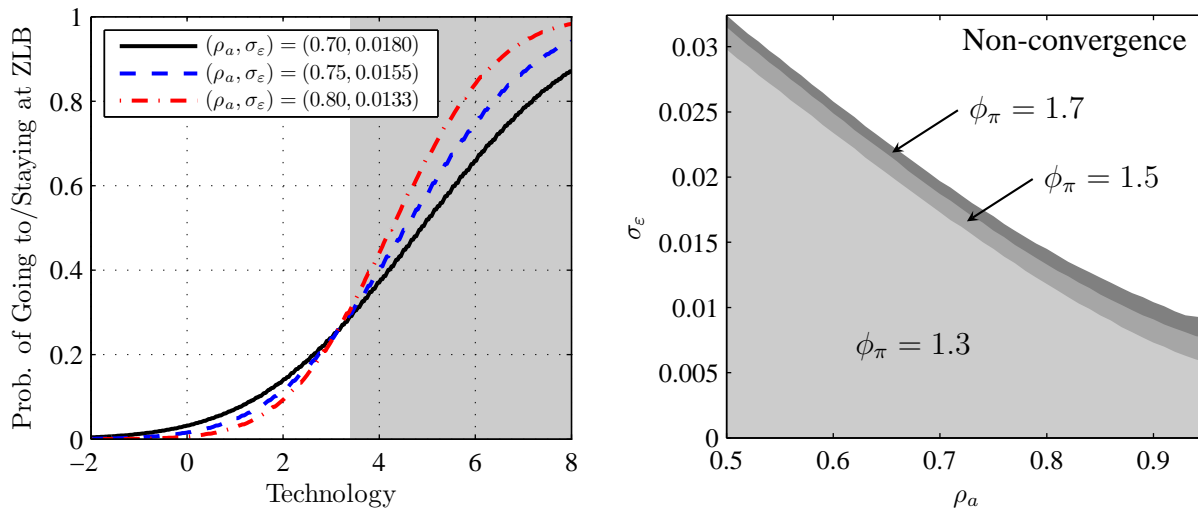
5.1 TECHNOLOGY SHOCKS In this section, technology evolves according to

$$a_t = \bar{a}(a_{t-1}/\bar{a})^{\rho_a} \exp(\varepsilon_t), \quad (11)$$

where $0 \leq \rho_a < 1$ and $\varepsilon_t \sim \mathbb{N}(0, \sigma_\varepsilon^2)$. The discount factor is constant ($\beta_t = \bar{\beta}$ for all t). We define $\sigma_a = \sigma_\varepsilon / (1 - \rho_a^2)^{1/2}$ as the standard deviation of (11). Positive technology shocks act as positive aggregate supply shocks. At high technology levels, firms' per unit marginal cost of production is low. Firms react by lowering their prices and raising their production. This causes deflation and, given a sufficiently high level of technology, the (net) nominal interest rate falls to zero according to the Taylor rule in (9). Thus, ZLB events are endogenous due to technology shocks.

Figure 5a plots (10) as a function of the technology state for three alternative parameterizations of (11). The shaded region corresponds to technology states where the ZLB binds, which begins when technology is 3.5 percent above its steady state. The three combinations of $(\rho_a, \sigma_\varepsilon)$ are chosen to keep the boundary of the ZLB region fixed. In technology states below the boundary, the probability on the vertical axis is the probability of going to the ZLB in the next quarter. In technology states above the boundary, it is the probability of staying at the ZLB. This figure demonstrates the tradeoff between the probability of hitting the ZLB and the average duration of ZLB events. As ρ_a increases and σ_ε decreases, it is less likely the ZLB will bind in technology states below the boundary and more likely the ZLB will continue to bind once the ZLB is hit.

The combinations of $(\rho_a, \sigma_\varepsilon)$ shown in figure 5a are *not* on the boundary of the convergence region in $(\rho_a, \sigma_\varepsilon)$ -space. The boundary of the ZLB region is a function of $(\rho_a, \sigma_\varepsilon)$, which affects the probabilities of going to and staying at the ZLB. Since ZLB events are endogenous due to (11),



(a) Probabilities of going to the ZLB, given the technology state and $\phi_\pi = 1.5$. Technology is in percent deviations from steady state. The shaded region corresponds to the states where the ZLB binds.

(b) Tradeoff between the technology process persistence and shock standard deviation across alternative monetary policy responses to inflation. The shaded regions correspond to the convergence regions.

Figure 5: Properties of the model where ZLB events arise endogenously due to technology shocks.

there is no way to map $(\rho_a, \sigma_\varepsilon)$ into equivalent (p_{11}, p_{22}) values and generate a picture equivalent to figure 2 (i.e., we cannot increase p_{22} by changing $(\rho_a, \sigma_\varepsilon)$ without altering p_{11}). Thus, fixing the boundary of the ZLB region offers the closest comparison to the Markov chain process in section 4.

Figure 5b shows that along the boundary of the convergence region (shaded), there is a clear tradeoff between the persistence of the technology process, ρ_a , and the standard deviation of the shock, σ_ε . As the persistence of the process increases, the standard deviation of the shock must decline to avoid a non-convergence region. This tradeoff reflects that ρ_a and σ_ε both impact the expected frequency and average duration of ZLB events, as figure 5a shows. Once again, the monetary policy response to inflation, ϕ_π , affects the size of the convergence region. For a given ρ_a , an increase in ϕ_π permits a larger σ_ε , as prices are more stable when the ZLB does not bind.

The fact that the parameters of the stochastic process impact the convergence region is significant, because these parameters do not affect convergence in linearized models, regardless of whether the ZLB is imposed. In models that impose a ZLB, it is common to linearize every equation in the equilibrium system, except for the Taylor rule, and assume ZLB events last for a predetermined duration with no probability of recurrence. This approach does not account for the expectational effects of going to and exiting the ZLB, which are critical for convergence.

Figure 6 compares the inflation rate decision rules across two parameterizations of (11), both of which are on the boundary of the convergence region in $(\rho_a, \sigma_\varepsilon)$ -space. The horizontal dashed line is the steady-state inflation rate ($\bar{\pi} = 1.005$). When the technology state equals the steady-state technology level ($\bar{a} = 1$), the deviations of the inflation rate from its steady-state value provide a measure of the expectational effect of hitting the ZLB. The shaded region represents values of the inflation rate where the ZLB binds. When σ_ε is relatively small (dashed line), the expectational effect is small because the likelihood of hitting the ZLB in expectation is also small. As σ_ε increases, and σ_a increases with it, the expectational effect of hitting the ZLB also increases.

When the ZLB binds, higher real interest rates reduce consumption and put downward pressure on inflation as firms respond to the lower demand. Thus, when there is a higher probability of going

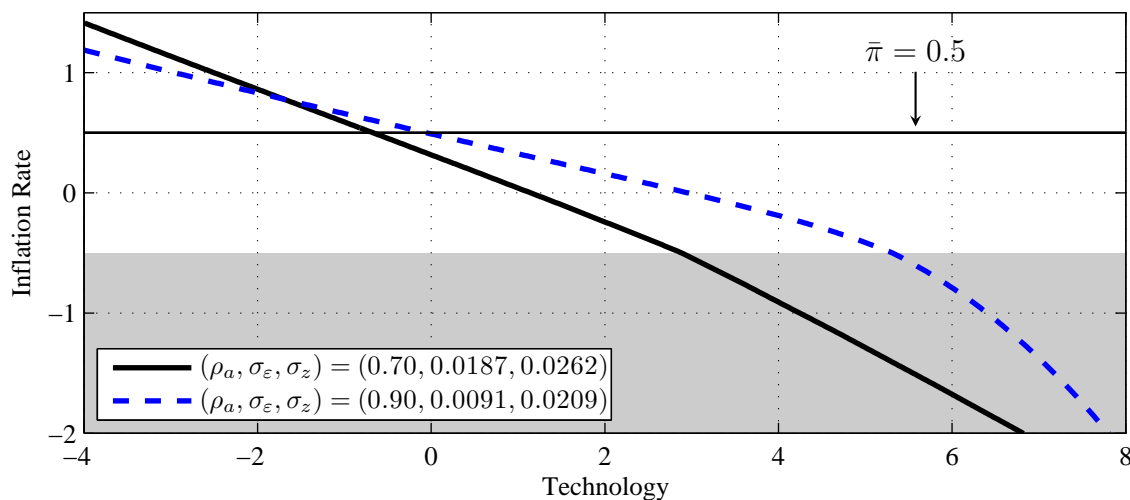


Figure 6: Comparison of the decision rules for $(\rho_a, \sigma_\varepsilon)$ combinations on the boundary of the convergence region ($\phi_\pi = 1.5$). Technology is in percent deviations from steady state. The solid horizontal line represents the steady-state gross inflation rate. The shaded region corresponds to the technology states where the ZLB binds.

to the ZLB (solid line), the slope of the inflation rate policy function is steeper. Since the downward pressure on inflation happens across the entire state space, it also influences where the ZLB first binds in the state space. For smaller standard deviations of (11), the probability of hitting the ZLB in expectation is smaller and the boundary of the ZLB region lies at a higher technology state. Unlike log-linearized models, where the calibration of the stochastic process has a much smaller effect on the decision rules, these results imply that changes in the calibration of the stochastic process can significantly impact the quantitative properties of the model.

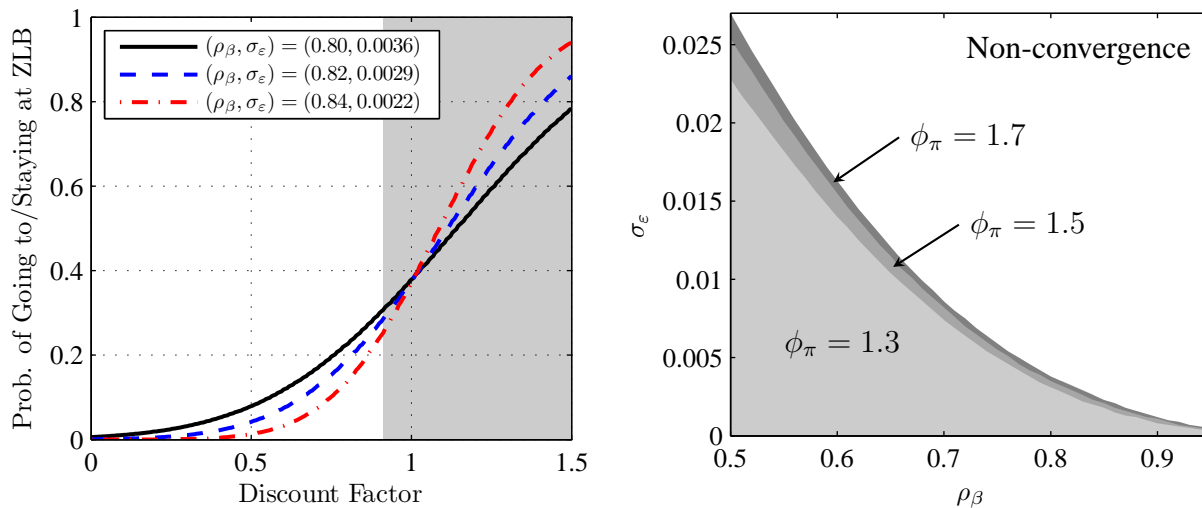
5.2 DISCOUNT FACTOR SHOCKS In this section, the discount factor evolves according to

$$\beta_t = \bar{\beta}(\beta_{t-1}/\bar{\beta})^{\rho_\beta} \exp(v_t), \quad (12)$$

where $0 \leq \rho_\beta < 1$ and $v_t \sim \mathcal{N}(0, \sigma_v^2)$. Technology is constant ($a_t = \bar{a}$ for all t). We define $\sigma_\beta = \sigma_v / (1 - \rho_\beta^2)^{1/2}$ as the standard deviation of (12). Positive discount factor shocks act as negative aggregate demand shocks. A high discount factor means that the household is more patient and elects increase leisure and to defer consumption to future periods. Firms respond to the lower demand by cutting output and reducing their prices. This causes deflation and, given a sufficiently high discount factor, the (net) nominal interest rate falls to zero according to the Taylor rule in (9). Thus, ZLB events are endogenous due to discount factor shocks.

Figure 7a reproduces figure 5 for three alternative parameterizations of the discount factor process given in (12). The shaded region corresponds to discount factor states where the ZLB binds, which begins when the discount factor is 0.9 percent above its steady-state value. Once again, there is a clear tradeoff between the expected frequency and average duration of ZLB events.

Figure 7b shows the convergence regions in (ρ_β, σ_v) -space. For a given persistence value, the discount factor process permits a much smaller shock size than the technology process. This is because the discount factor directly affects the household's willingness to intertemporally substitute, which is critical for convergence since it affects expected inflation. As an example, the maximum shock size is only 0.0003 when $\rho_\beta = 0.95$. This is significant because estimates of this parameter



(a) Probabilities of going to the ZLB, given the discount factor state and $\phi_\pi = 1.5$. The discount factor is in percent deviations from steady state. The shaded region corresponds to the states where the ZLB binds.

(b) Tradeoff between the discount factor process persistence and shock standard deviation across alternative monetary policy responses to inflation. The shaded regions correspond to the convergence regions.

Figure 7: Properties of the model where ZLB events arise endogenously due to discount factor shocks.

using a log-linear model without a ZLB constraint are outside of this region. The data prefers highly persistent shocks (i.e., $\rho_\beta > 0.95$) with a standard deviation that is over four times the maximum value inside the convergence region. While the model is slightly different, the estimates of the constrained nonlinear model in Gust et al. (2013) are also well outside of the convergence region. They estimate that $\rho_\beta = 0.88$ and $\sigma_v = 0.0025$, which may be possible in our model with a persistent Taylor rule or more price stickiness. The data also prefers highly persistent technology shocks, but it does not pose as serious a problem for estimation because the constrained model permits large shocks. When $\rho_a = 0.95$ and $\phi_\pi = 1.5$, the maximum shock size is 0.75 percent.

Figure 8 plots the decision rules for inflation. The slope is steeper (more negative) when the discount factor is more persistent. In discount factor states where the ZLB does (does not) bind, inflation is lower (higher). At the ZLB, higher persistence means the household expects relatively higher consumption growth. Since the nominal interest rate does not respond to inflation, the only way for the real interest rate to rise and for the bond market to clear is if inflation falls sharply. The expectational effect of the ZLB in discount factor states where the ZLB does not bind also drives down inflation. In states further from ZLB region, the expectational effect is weaker and the higher demand associated with a lower, more persistent discount factor increases inflation. Once again, these results show that even small changes in the parameterization of the exogenous driving process significantly affect the decision rules, and hence the quantitative properties of the model.

6 CONCLUSION

This paper demonstrates that the boundary of the convergence region imposes a clear tradeoff between the expected frequency and average duration of episodes at the ZLB, regardless of whether ZLB events arise exogenously or endogenously. This tradeoff is critical for at least three reasons. First, even though the Taylor principle does not hold at the ZLB, it shows that central banks can still pin down prices when the nominal interest rate is pegged at its ZLB, so long as households

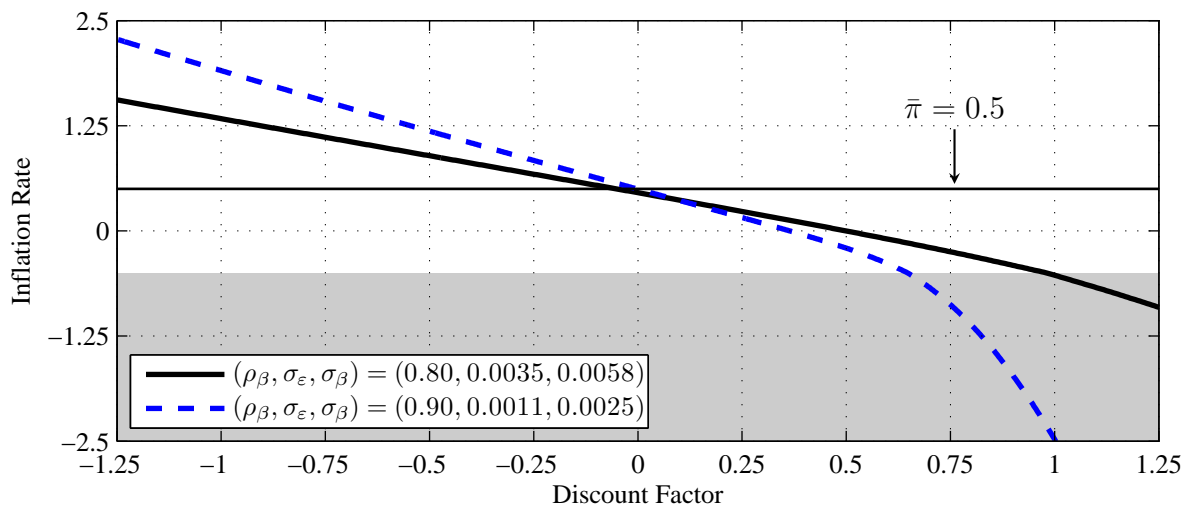


Figure 8: Comparison of the decision rules for (ρ_β, σ_v) combinations on the boundary of the convergence region ($\phi_\pi = 1.5$). The discount factor is in percent deviations from steady state. The solid horizontal line represents the steady-state gross inflation rate. The shaded region corresponds to the discount factor states where the ZLB binds.

have a strong enough expectation of returning to a regime where the central bank aggressively responds to inflation. Second, it imposes an important constraint on the parameter space that the econometrician must account for when estimating the fully nonlinear model. Third, it implies that small changes in the parameters of stochastic processes significantly impact the decision rules and the state at which the ZLB first binds. This means accurately calibrating or estimating the parameters of the exogenous driving processes is particularly important for analysis at the ZLB.

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A NUMERICAL ALGORITHM

A formal description of the numerical algorithm begins by writing the model compactly as

$$\mathbb{E}[f(\mathbf{v}_{t+1}, \mathbf{w}_{t+1}, \mathbf{v}_t, \mathbf{w}_t) | \Omega_t] = 0,$$

where f is vector-valued function that contains the equilibrium system, \mathbf{v} is a vector of exogenous variables, \mathbf{w} is a vector of endogenous variables, and $\Omega_t = \{M, P, \mathbf{z}_t\}$ is the household's information set in period t , which contains the structural model, M , its parameters, P , and the state vector, \mathbf{z} . In the model where ZLB events are exogenous, $\mathbf{v} = \mathbf{z} = (s)$. When ZLB events are endogenous due to technology shocks $\mathbf{v} = (a, \varepsilon)$ and $\mathbf{z} = a$ and when ZLB events are endogenous due to discount factor shocks $\mathbf{v} = (\beta, v)$ and $\mathbf{z} = \beta$. In all models, $\mathbf{w} = (c, \pi, y, n, w, r)$.

Policy function iteration approximates the vector of decision rules, Φ , as a function of the state vector, \mathbf{z} . The time-invariant decision rules for the exogenous model are

$$\underbrace{\Phi(\mathbf{z}_t)}_{\text{True RE Solution}} \approx \underbrace{\hat{\Phi}(\mathbf{z}_t)}_{\text{Approximating Function}}.$$

We choose to iterate on $\Phi = (c, \pi)$ so that we can easily solve for future variables that enter the household's expectations using f . Each continuous state variable in \mathbf{z} is discretized into N^d points, where $d \in \{1, \dots, D\}$ and D is the dimension of the state space. The discretized state space is represented by a set of unique D -dimensional coordinates (nodes). In general, we set the bounds of continuous stochastic state variables to encompass 99.999 percent of the probability mass of the distribution. We specify 1001 grid points for each continuous state variable and use the maximum number of Gauss-Hermite nodes (66) for each continuous shock. These techniques minimize extrapolation and ensure that the location of the kink in the decision rules is accurate.

The following outline summarizes the policy function algorithm we employ. Let $i \in \{0, \dots, I\}$ index the iterations of the algorithm and $n \in \{1, \dots, \prod_{d=1}^D N^d\}$ index the nodes.

1. Obtain initial conjectures for the approximating functions, \hat{c}_0 and $\hat{\pi}_0$, on each node, from the log-linear model without the ZLB imposed. We use `gensys.m` to obtain these conjectures.
2. For $i \in \{1, \dots, I\}$, implement the following steps:
 - (a) On each node, solve for $\{y_t, r_t, n_t, w_t\}$ given $\hat{c}_{i-1}(\mathbf{z}_t^n)$ and $\hat{\pi}_{i-1}(\mathbf{z}_t^n)$ with the ZLB imposed.
 - (b) In the endogenous model, linearly interpolate $\{c_{t+1}, \pi_{t+1}\}$ given $\{\varepsilon_{t+1}^m\}_{m=1}^M$. Each of the M values ε_{t+1}^m are Gauss-Hermite quadrature nodes. We use Gauss-Hermite quadrature to numerically integrate, since it is very accurate for normally distributed shocks. We use piecewise linear interpolation to approximate future variables that show up in expectation, since this approach more accurately captures the kink in the decision rules than continuous functions such as cubic splines or Chebyshev polynomials.⁹

⁹Aruoba and Schorfheide (2013) use a linear combination of two Chebyshev polynomials—one that captures the dynamics when the ZLB binds and one that captures the dynamics when the Taylor principle holds. While this approach is more accurate than using one Chebyshev polynomial, there is no guarantee that it will accurately locate the kink. Moreover, Chebyshev polynomials can lead to large approximation errors due to extrapolation. With linear interpolation, a dense state space will lead to more predictable extrapolation and more accurately locate the kink [Richter et al. (2013)].

- (c) We use the nonlinear solver, `csolve.m`, to minimize the Euler equation errors. On each node, numerically integrate to approximate the expectation operators,

$$\mathbb{E} [f(\mathbf{x}_{t+1}^k, \mathbf{x}_t^n) | \Omega_t] \approx \frac{1}{\sqrt{\pi}} \sum_{j=1}^2 p_{jk} f(\mathbf{x}_{t+1}^k, \mathbf{x}_t^n), \quad (\text{Exogenous ZLB Model})$$

$$\mathbb{E} [f(\mathbf{x}_{t+1}^m, \mathbf{x}_t^n) | \Omega_t] \approx \frac{1}{\sqrt{\pi}} \sum_{m=1}^M f(\hat{\mathbf{x}}_{t+1}^m, \hat{\mathbf{x}}_t^n) \phi(\varepsilon_{t+1}^m), \quad (\text{Endogenous ZLB Model})$$

where $\mathbf{x} \equiv (\mathbf{v}, \mathbf{w})$, $p_{jk} = \Pr(s_{t+1} = k | s_t^n = j)$, and ϕ are the respective Gauss-Hermite weights. The superscripts on \mathbf{x} indicate which realizations of the state variables are used to compute expectations. The nonlinear solver searches for $\hat{c}_i(\mathbf{z}_t^n)$ and $\hat{\pi}_i(\mathbf{z}_t^n)$ so that the Euler equation errors are less than 1^{-4} on each node.

3. Define $\text{maxdist}_i \equiv \max\{|\hat{c}_i - \hat{c}_{i-1}|, |\hat{\pi}_i - \hat{\pi}_{i-1}|\}$. Repeat the steps in [item 2](#) until one of the following conditions is satisfied.
 - If for all n , $\text{maxdist}_i < 1^{-13}$ for 10 consecutive iterations, then the algorithm converged to a bounded MSV solution. Since the state is composed of only exogenous variables, the solution is bounded so long as the decisions rules are positive and finite.
 - Otherwise, we say the algorithm is non-convergent for one of the following reasons:
 - $i = I = 500,000$ (Algorithm times out)
 - For all n and any i , $\hat{\pi}_i < .5$, or for any n , $\hat{c}_i < 0$ (Approximating functions drift)
 - Define $\text{dir}_i = \text{maxdist}_i - \text{maxdist}_{i-1}$. For all n , $\text{dir}_i \geq 0$ and $\text{dir}_i - \text{dir}_{i-1} \geq 0$ for 100 consecutive iterations (Algorithm diverges)

To provide evidence that each MSV solution is locally unique, we randomly perturb the converged decision rules and check that the algorithm converges back to the same solution.