The Matching Function and Nonlinear Business Cycles*

Joshua Bernstein

Alexander W. Richter   Nathaniel A. Throckmorton

July 12, 2022

Abstract

The Cobb-Douglas matching function is ubiquitous in labor search and matching models, even though it imposes a constant matching elasticity that is inconsistent with recent empirical evidence. To examine the implications of this discrepancy, this paper first uses a general constant returns to scale matching function to derive conditions that determine how the cyclicality of the matching elasticity amplifies or dampens nonlinear labor market dynamics. It then shows that modest cyclical variation in the matching elasticity, in line with the recent estimates, generates large differences in higher-order moments and has significant effects on optimal policy.

Keywords: Matching Function; Matching Elasticity; Nonlinear; Finding Rate; Unemployment
JEL Classifications: E24; E32; E37; J63; J64

*Bernstein, Department of Economics, Indiana University, 100 S. Woodlawn, Bloomington, IN 47405 (jmbernst@iu.edu); Richter, Research Department, Federal Reserve Bank of Dallas, 2200 N Pearl Street, Dallas, TX 75201 (alex.richter@dal.frb.org); Throckmorton, Department of Economics, William & Mary, P.O. Box 8795, Williamsburg, VA 23187 (nat@wm.edu). We thank Nicolas Petrosky-Nadeau for his discussion at the 2022 American Economic Association Meeting. We also thank Domenico Ferraro, Emily Moschini, and Francesco Zanetti for comments that helped improve the paper. This work was supported by computational resources provided by the BigTex High Performance Computing Group at the Federal Reserve Bank of Dallas. The views expressed in this paper are our own and do not necessarily reflect the views of the Federal Reserve Bank of Dallas or the Federal Reserve System.

1 INTRODUCTION

The matching function—the mapping from job seekers and vacancies into matches—is a core component of search and matching models. In particular, the elasticity of matches with respect to vacancies, which we refer to as the matching elasticity, is a key object in empirical and structural work. While it is common to use a Cobb-Douglas matching function in structural models, its constant matching elasticity is a knife-edge case that is unlikely to hold empirically. By relaxing that assumption, this paper shows that modest variation in the matching elasticity has significant positive and normative implications for the nonlinear properties of the search and matching model.

To motivate our analysis, we first review the extensive empirical literature that estimates the matching elasticity. Although most of this work imposes the typical Cobb-Douglas specification, the wide range of estimates suggests that a fixed matching elasticity does not provide the best description of the data. Furthermore, Lange and Papageorgiou (2020) non-parametrically estimate the matching function and find support for a procyclical elasticity that fluctuates between 0.15 and 0.3. This motivates us to characterize the nonlinear effects of a general constant returns to scale matching function without *a priori* restrictions on the mean or cyclicality of the matching elasticity.

A simple example demonstrates how cyclical variation in the matching elasticity affects nonlinear labor market dynamics. Consider a positive productivity shock that causes firms to post more vacancies, increasing match creation and the job finding rate. A procyclical matching elasticity amplifies this transmission, while a countercyclical elasticity dampens it. The opposite applies to the transmission of a negative shock, which is dampened by a procyclical elastcity and amplified by a countercyclical elasticity. Therefore, a cyclical matching elasticity will asymmetrically amplify or dampen the transmission of positive and negative shocks, creating a source of nonlinear dynamics.

Our analytical results uncover simple conditions that characterize the strength of the asymmetry and only depend on the matching elasticity and the elasticity of substitution between vacancies and job seekers. Importantly, we show that the elasticity of substitution governs the cyclicality of the matching elasticity. Higher substitutability dampens the diminishing returns to vacancy creation. When this effect is sufficiently strong, the matching elasticity is increasing in vacancy creation and procyclical. Therefore, higher substitutability in the matching process tends to amplify positive shocks and dampen negative shocks, while lower substitutability leads to the opposite asymmetry.

To quantify the mechanism, we impose the constant elasticity of substitution (CES) functional form, which nests the Cobb-Douglas specification. We find that modest cyclical variation in the matching elasticity, in line with Lange and Papageorgiou's estimates, generates large differences in higher-order business cycle moments. For example, when holding the standard deviation of the unemployment rate fixed, switching from countercyclical to procyclical variation lowers the skewness of the unemployment rate from 2.37 to 0.29, offsetting the asymmetry in the law of motion for employment and nearly eliminating the nonlinear labor market dynamics emphasized in the literature.

Finally, we derive the normative implications of a general matching function, thus extending the well-known results for the Cobb-Douglas specification. Away from this knife-edge case where the matching elasticity is constant, we show that the cyclicality of the matching elasticity qualitatively affects the cyclicality of the vacancy tax that alleviates the externalities endemic to the frictional matching process. In addition, the differences in the nonlinear unemployment dynamics that we document across different matching functions transmit to consumption and hence to cyclical movements in the efficient real interest rate, which is a key ingredient of optimal monetary policy design. Understanding the true matching function is crucial for the conduct of optimal policy interventions.

Related Literature Our contribution is to analytically uncover a general mechanism through which the matching function generates nonlinearities in the search and matching model, and to quantify its positive and normative implications. Our results complement a growing literature that uses the search and matching model to analyze business cycle asymmetries and nonlinearities (e.g., Abbritti and Fahr, 2013; Dupraz et al., 2019; Ferraro, 2018; Ferraro and Fiori, 2021; Petrosky-Nadeau and Zhang, 2017; Petrosky-Nadeau et al., 2018; Pizzinelli et al., 2020). While these papers use a specific matching function and focus on other mechanisms, we show the matching function itself is a powerful source of nonlinear dynamics. In light of the uncertainty surrounding the true matching function and the recent empirical estimates, our results emphasize the need to consider alternative specifications when assessing a model's ability to produce nonlinear features of the data.

The matching function specification also has significant normative implications. Hairault et al. (2010) and Jung and Kuester (2011) study how nonlinearities in the search and matching model affect the welfare cost of business cycles. While they derive conditions that determine how the shape of the job finding rate function affects welfare, they do not uncover the underlying mechanism, which we show depends on offsetting effects that the Cobb-Douglas restriction obscures. Several papers have also examined how nonlinear search and matching frictions affect optimal policy, but only under a Cobb-Douglas matching function (e.g., Arseneau and Chugh, 2012; Faia, 2009; Jung and Kuester, 2015; Lepetit, 2020). We show the matching function itself has meaningful effects on the efficiency-restoring fiscal policies and the responses of the efficient real interest rate to shocks.

Our analysis also sheds light on the properties of the matching function introduced by Den Haan et al. (2000), which is used in influential papers such as Hagedorn and Manovskii (2008) and Petrosky-Nadeau et al. (2018).[1] While that specification has been used interchangeably with the Cobb-Douglas matching function,[2] we show they have different nonlinear properties. In contrast

[1] Bernstein et al. (2021), Ferraro (2018), Hashimzade and Ortigueira (2005), and Petrosky-Nadeau and Zhang (2017, 2021) also use this matching function. Stevens (2007) provides a microfoundation for such a matching function.

[2] When comparing the Cobb-Douglas and Den Haan et al. (2000) matching functions Petrosky-Nadeau and Wasmer (2017) say the "business cycle moments of the model using either functional form are similar." The justification for using the Den Haan et al. (2000) specification is that it restricts the job filling and job finding rates to the unit interval.

with the estimates in Lange and Papageorgiou (2020), the Den Haan et al. (2000) specification generates countercyclical variation in the matching elasticity that introduces concavity in the job finding rate and amplifies nonlinear labor market dynamics relative to the Cobb-Douglas specification.

Outline The rest of the paper proceeds as follows. Section 2 provides an overview of the key properties of the matching function and the empirical estimates of the matching elasticity. Section 3 lays out our search and matching model. Section 4 derives a closed-form solution and characterizes the sources of nonlinearity. Section 5 quantifies the nonlinearities and their effects on labor market dynamics. Section 6 shows the normative implications of the matching function. Section 7 concludes.

2 OVERVIEW OF MATCHING FUNCTIONS

To motivate our analytical and quantitative exercises, we briefly discuss some useful theoretical properties of matching functions and review the associated empirical literature that estimates them. We consider matching functions of the form $\mathcal{M}(u_t^s, v_t)$, where u_t^s measures the search effort of job seekers (often counts of unemployed workers) and v_t measures the recruitment effort of employers (often counts of vacancy postings). Throughout, we assume $\mathcal{M}(u_t^s, v_t)$ is strictly increasing, strictly concave, and twice differentiable in both arguments, and exhibits constant returns to scale (see Petrongolo and Pissarides (2001) for an overview of the evidence supporting constant returns).

A key object of theoretical and empirical interest is the elasticity of matches with respect to vacancies, which we denote by $\epsilon_t = \mathcal{M}_v(u_t^s, v_t)v_t/\mathcal{M}(u_t^s, v_t)$ and refer to as the matching elasticity. We note that due to constant returns to scale, the matching elasticity depends only on labor market tightness, $\theta_t = v_t/u_t^s$, and lies in the unit interval: $\epsilon(\theta_t) = \mathcal{M}_v(1, \theta_t)\theta_t/\mathcal{M}(1, \theta_t) \in (0, 1)$. Although some papers in the literature focus on the matching elasticity with respect to search effort, constant returns to scale also implies that $\mathcal{M}_u(u_t^s, v_t)u_t^s/\mathcal{M}(u_t^s, v_t) = 1 - \mathcal{M}_v(u_t^s, v_t)v_t/\mathcal{M}(u_t^s, v_t)$.

The goal of this paper is to uncover how the statistical properties of the matching elasticity (e.g., its mean, standard deviation, and cyclicality) affect nonlinear dynamics, given their recent emphasis in the search and matching literature. The following result establishes a benchmark that applies when restricting attention to linear models. All of the proofs are contained in Appendix A.

Proposition 1. *To first order, any constant returns to scale matching function is equivalent to a Cobb-Douglas matching function, $\mathcal{M}(u_t^s, v_t) = \phi(u_t^s)^{1-\bar{\epsilon}}v_t^{\bar{\epsilon}}$, where $\bar{\epsilon}$ is a fixed matching elasticity.*

The Cobb-Douglas matching function is a common assumption in business cycle research.[3] It imposes that the matching elasticity is invariant to labor market conditions. Proposition 1 shows that when we restrict attention to linear dynamics, this assumption is without loss of generality. Intuitively, in a linear model, only the value of the matching elasticity in the deterministic steady state affects dynamics. This value can be set as a parameter of a Cobb-Douglas matching function.

[3] See, for example, Ljungqvist and Sargent (2017), Hall and Milgrom (2008), Pissarides (2009), and Shimer (2005).

In this paper, we depart from the special linear case and shed light on the higher-order positive and normative consequences of the matching function. To lay the foundations, we first establish how the matching elasticity in general varies with labor market conditions, as measured by labor market tightness. To do so, it is useful to define the elasticity of substitution between vacancies and job seekers, $\sigma_t = \frac{d\ln(v_t/u_t^s)}{d\ln(\mathcal{M}_u(u_t^s,v_t)/\mathcal{M}_v(u_t^s,v_t))} \in (0,\infty)$, which also only depends on labor market tightness due to constant returns to scale in the matching function: $\sigma(\theta_t) = \frac{d\ln\theta_t}{d\ln(\mathcal{M}_u(1,\theta_t)/\mathcal{M}_v(1,\theta_t))}$.

Proposition 2. *The matching elasticity, $\epsilon_t = \epsilon(\theta_t)$, is increasing in θ_t when $\sigma_t = \sigma(\theta_t) > 1$, constant when $\sigma_t = 1$, and decreasing when $\sigma_t < 1$.*

Recall that the matching elasticity is the marginal product of labor market tightness divided by the average product: $\epsilon(\theta_t) = \mathcal{M}_v(1,\theta_t)/(\mathcal{M}(1,\theta_t)/\theta_t)$. The effects of tightness on each term drive the matching elasticity in opposite directions. First, the average product is decreasing in tightness because a 1% increase in tightness yields a less than 1% increase in matches. This causes the matching elasticity to increase. Second, the marginal product is decreasing in tightness due to diminishing returns to vacancy creation. This causes the matching elasticity to decrease. The dominant effect depends on how quickly the marginal product declines, which is governed by the elasticity of substitution. When $\sigma(\theta_t) > 1$, high substitutability between vacancies and job seekers slows the decline in the marginal product, so the first effect dominates and $\epsilon(\theta_t)$ is increasing in tightness.

Proposition 2 uncovers a tight relationship between variation in the matching elasticity and the elasticity of substitution that applies to a general matching function. We obtain further structure if we are willing to impose a functional form on the matching function. For example, it is common to assume the matching function is Cobb-Douglas, which is a special case of the general CES family,

$$\mathcal{M}(u_t^s, v_t^s) = \begin{cases} \phi\left(\vartheta(u_t^s)^{(\sigma-1)/\sigma}(1-\vartheta)v_t^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)} & \sigma \neq 1, \\ \phi(u_t^s)^{\vartheta}v_t^{1-\vartheta} & \sigma = 1, \end{cases}$$

where $\phi > 0$ is matching efficiency and $\vartheta \in (0,1)$ is the importance of job seekers. Under this specification, the elasticity of substitution, $\sigma(\theta_t) = \sigma$, is fixed and we can strengthen Proposition 2.

Corollary 1. *Suppose the matching function is from the CES family. Then $\sigma > 1$ implies $\epsilon'(\theta_t) > 0$, $\sigma = 1$ implies $\epsilon'(\theta_t) = 0$, and $\sigma < 1$ implies $\epsilon'(\theta_t) < 0$ for all $\theta_t > 0$.*

Since tightness is procyclical in the data and in search and matching models, the choice of σ globally affects the cyclicality of the matching elasticity. When $\sigma = 1$, the matching function is Cobb-Douglas, and the matching elasticity is constant, $\epsilon_t = \bar{\epsilon} = 1 - \vartheta$. Away from this special case, higher substitutability ($\sigma > 1$) generates procyclical variation in the matching elasticity, while lower substitutability ($\sigma < 1$) implies countercyclical variation. Our analytical and quantitative exercises will shed light on how the cyclicality of ϵ_t translates into nonlinear labor market dynamics.

Author(s)	Method(s)	Sample	Parameter Estimates	
Cobb-Douglas			$\bar{\epsilon}$	
Blanchard and Diamond (1989)	OLS, AR1 residual	1968-1981	0.54	
Bleakley and Fuhrer (1997)	OLS with breakpoints	1979-1993	0.31-0.35	
Shimer (2005)	OLS, AR1 residual	1951-2003	0.28	
Hall (2005)	OLS	2000-2002	0.77	
Rogerson and Shimer (2011)	OLS, multiplicative noise	2001-2009	0.42	
Michaillat and Saez (2021)	OLS with breakpoints	1951-2019	0.51-0.61	
Cobb-Douglas with endogeneity correction			$\bar{\epsilon}$	
Borowczyk-Martins et al. (2013)	GMM IV	2000-2012	0.70	
Şahin et al. (2014)	OLS, GMM IV, varied data	2001-2012	0.24-0.66	
Barnichon and Figura (2015)	GMM IV	1968-2007	0.34	
Sedláček (2016)	OLS with non-unemployed	2000-2013	0.24	
Hall and Schulhofer-Wohl (2018)	OLS with aggregation	2001-2013	0.35	
CES			$\bar{\epsilon}$	σ
Blanchard and Diamond (1989)	NLS, AR1 residual	1968-1981	0.54	0.74
Shimer (2005)	NLS, AR1 residual	1951-2003	0.28	1.06
Şahin et al. (2014)	GMM IV, varied data	2001-2012	0.24-0.66	0.9-1.2
Non-parametric				
Lange and Papageorgiou (2020)	Non-parametric	2001-2017	$\epsilon_t \in (0.15, 0.3)$	Procyclical

Table 1: Empirical estimates of the matching function. $\bar{\epsilon}$ is a fixed matching elasticity, σ is the elasticity of substitution, and ϵ_t is a time-varying matching elasticity. *Cobb-Douglas* lists papers that impose the Cobb-Douglas functional form. *Cobb-Douglas with endogeneity correction* lists papers that account for endogeneity and impose the Cobb-Douglas functional form at the aggregate or job status level. *CES* lists papers that use the CES functional form. *Non-parametric* lists papers that do not impose a functional form.

Empirical Evidence Table 1 summarizes the empirical literature that estimates either a fixed or time-varying matching elasticity. For brevity and to permit a cleaner comparison of the estimates, we focus on studies that use U.S. data and impose constant returns to scale in the matching function.

Early work used aggregate data on hires, vacancies, and unemployment to directly estimate a log-linear matching function with OLS. Following the logic of Proposition 1, this approach implicitly assumed a Cobb-Douglas matching function and estimated the fixed matching elasticity. Due to differences in data sources and samples, the estimates ranged from around 0.3 in Bleakley and Fuhrer (1997) and Shimer (2005) up to 0.77 in Hall (2005). Furthermore, estimates based on data from the more recent JOLTS survey (Hall, 2005; Rogerson and Shimer, 2011) are higher than past estimates based on CPS flows data (Bleakley and Fuhrer, 1997) or the Shimer (2005) method. More recently, Michaillat and Saez (2021) develop a different approach in which they first estimate the elasticity of vacancies with respect to unemployment using OLS with breakpoints and then use a search and matching model to solve for the matching elasticity, which ranges from 0.51 to 0.61.

More recent work developed methods to deal with potential endogeneity due to unobserved variation in matching efficiency (the ϕ term in the Cobb-Douglas specification above), either by using instruments (Borowczyk-Martins et al., 2013) or by exploiting heterogeneity in job seekers (Hall and Schulhofer-Wohl, 2018). In addition, Sedláček (2016) proposed a latent-variable strategy to deal with unobserved job search by non-unemployed workers. These papers maintained the Cobb-Douglas assumption at either the aggregate or job status level and generated estimates in the same range as the estimates that did not correct for endogeneity. The broad range of estimates is again at least partially due to different data choices, with higher estimates generated by JOLTS data (Borowczyk-Martins et al., 2013; Şahin et al., 2014) and a lower estimates generated by CPS flows data (Barnichon and Figura, 2015) or industry-level hires from CPS data (Şahin et al., 2014).

A few papers relax the Cobb-Douglas assumption. Imposing the CES functional form, Blanchard and Diamond (1989) estimated an elasticity of substitution of 0.74. More recently, Shimer (2005) and Şahin et al. (2014) obtained estimates closer to 1 but with larger standard errors, indicating weak identification. Given Corollary 1, this suggests that the cyclicality of the matching elasticity is highly uncertain. Finally, Lange and Papageorgiou (2020) propose a non-parametric identification strategy that deals with potential endogeneity. They estimate a procyclical elasticity that fluctuates between 0.15 and 0.3, which is consistent with a CES matching function with $\sigma > 1$.

Outlook There is considerable uncertainty surrounding the matching elasticity, even when it is assumed to be fixed. Among papers relaxing that assumption, there is additional uncertainty about the elasticity of substitution between vacancies and job seekers and the cyclicality of the matching elasticity. The most recent and most general econometric specification finds that the matching elasticity is procyclical, in contrast with the Cobb-Douglas matching function. The lack of consensus and implausibility of a fixed matching elasticity shows why it is important to investigate the implications of a time-varying matching elasticity. As we will show, even modest variation has significant implications, which motivates empirical work that can provide greater clarity on the matching function.

3 ENVIRONMENT

To cleanly demonstrate our results, we use a textbook search and matching model. The one exception is that we use a general constant returns to scale matching function, rather than assuming a particular functional form. Each period denotes 1 month and the population (equal to the labor force) is normalized to unity. Business cycles are driven by shocks to labor productivity, a_t, which follows

$$a_{t+1} = \bar{a} + \rho_a(a_t - \bar{a}) + \sigma_a \varepsilon_{a,t+1},\ 0 \leq \rho_a < 1,\ \varepsilon_a \sim \mathbb{N}(0,1). \tag{1}$$

Search and Matching Entering period t, there are n_{t-1} employed workers and $u_{t-1} = 1 - n_{t-1}$ unemployed job seekers. In period t, firms post v_t vacancies, so the number of matches is given by

$$m_t = \min\{\mathcal{M}(u_{t-1}, v_t), u_{t-1}, v_t\}, \tag{2}$$

where $\mathcal{M}$ is a constant returns to scale matching function that satisfies the assumptions in Section 2.

Given the number of matches, the job finding rate, job filling rate, and laws of motion satisfy

$$f_t = m_t/u_{t-1}, \tag{3}$$

$$q_t = m_t/v_t, \tag{4}$$

$$n_t = (1-\bar{s})n_{t-1} + f_t u_{t-1}, \tag{5}$$

$$u_t = u_{t-1} + \bar{s}n_{t-1} - f_t u_{t-1}, \tag{6}$$

where $u_t = 1 - n_t$, $\bar{s} \in (0,1)$ is the exogenous separation rate, and (2) ensures that $f_t, q_t \in [0,1]$.

Firms A representative firm chooses vacancies and employment $\{v_t, n_t\}$ to solve

$$V_t = \max_{v_t, n_t} a_t n_t - w_t n_t - \kappa v_t + E_t[x_{t+1} V_{t+1}]$$

subject to $n_t = (1-\bar{s})n_{t-1} + q_t v_t$ and $v_t \geq 0$, where $\kappa > 0$ is the vacancy posting cost, w_t is the wage rate, and E_t is an expectation operator conditional on time-t information. The representative household's pricing kernel is $x_{t+1} = \beta(c_t/c_{t+1})^{\gamma}$, where c_t is consumption, $\beta \in (0,1)$ is the discount factor, and $\gamma \geq 0$ is the coefficient of relative risk aversion.[4] The optimality conditions imply

$$\tfrac{\kappa - \lambda_{v,t}}{q_t} = a_t - w_t + (1-\bar{s})E_t\left[x_{t+1}\tfrac{\kappa - \lambda_{v,t+1}}{q_{t+1}}\right], \tag{7}$$

$$\lambda_{v,t} v_t = 0, \quad \lambda_{v,t} \geq 0, \tag{8}$$

where $\lambda_{v,t}$ is the multiplier on the non-negativity constraint $v_t \geq 0$. Condition (7) sets the marginal cost of hiring, $(\kappa - \lambda_{v,t})/q_t$, equal to the marginal benefit of hiring, which consists of the flow profits from the match, $a_t - w_t$, plus the savings from not having to post the vacancy in the future.

Wages As is common in the search and matching literature, wages are determined through Nash bargaining between employed workers and the firm. Following the steps in Appendix A, we obtain

$$w_t = \eta(a_t + \kappa E_t[x_{t+1}(v_{t+1}/u_t)]) + (1-\eta)b, \tag{9}$$

where $\eta \in (0,1)$ is the worker's bargaining power and $b > 0$ is the flow value of unemployment.

Equilibrium The aggregate resource constraint is given by

$$c_t + \kappa v_t = a_t n_t. \tag{10}$$

An equilibrium is infinite sequences of quantities $\{c_t, n_t, u_t, v_t, m_t, f_t, q_t\}_{t=0}^{\infty}$, prices $\{w_t, \lambda_{v,t}\}_{t=0}^{\infty}$, and productivity $\{a_t\}_{t=1}^{\infty}$ that satisfy (1)-(10) given the initial state $\{n_{-1}, a_{-1}\}$ and shocks $\{\varepsilon_{a,t}\}_{t=0}^{\infty}$.

[4]When households are risk averse ($\gamma > 0$), we follow the business cycle literature and assume there is perfect consumption insurance for employed and unemployed workers (Andolfatto, 1996; Den Haan et al., 2000; Merz, 1995).

4 Analytical Results

This section analytically characterizes how the matching function affects the nonlinear dynamics of the job finding and unemployment rates. Convexity or concavity of the job finding rate implies that the productivity shock transmission is asymmetric. If the finding rate is convex ($f''(a_t) > 0$), then a positive productivity shock at a_t will have a larger impact on the finding rate than a negative shock, creating positive skewness. Conversely, if the finding rate is concave ($f''(a_t) < 0$), then a negative shock will have a larger impact than a positive shock, creating negative skewness. The skewness in the job finding rate transmits to skewness in the unemployment rate through its law of motion.

Our results show that convexity or concavity depends on σ_t, which controls the cyclicality of ϵ_t according to Proposition 2. In particular, the job finding rate is convex if the matching elasticity is sufficiently procyclical. Intuitively, a procyclical matching elasticity increases the transmission of vacancies to matches when productivity increases, which amplifies the finding rate response and generates convexity. Likewise, the finding rate is concave when the matching elasticity is sufficiently countercyclical, as positive shock responses are dampened by a falling matching elasticity.

4.1 Model Solution To solve the model analytically, we make two simplifying restrictions.

Assumption 1. $\gamma = \eta = 0$.

These conditions imply that workers are risk neutral and have zero bargaining weight, so wages are sticky with $w_t = b$ (Hall, 2005).[5] We relax these restrictions in Section 5 for our quantitative exercises. Given these conditions, we obtain an analytical expression for the marginal cost of hiring.

Proposition 3. *Under Assumption 1, the marginal cost of hiring follows the stochastic process*

$$(\kappa - \lambda_{v,t})/q_t = \delta_0 + \delta_1(a_t - \bar{a}), \tag{11}$$

where

$$\delta_0 = \frac{\bar{a} - b}{1 - \beta(1 - \bar{s})} > 0, \qquad \delta_1 = \frac{1}{1 - \beta(1 - \bar{s})\rho_a} > 0,$$

and $\lambda_{v,t} > 0$ *implies* $q_t = 1$.

In (11), δ_0 is the steady-state marginal cost of hiring, while δ_1 is the response of marginal cost to changes in productivity. Intuitively, δ_1 is increasing in the persistence of the productivity shock ρ_a.[6]

In the data, job finding and job filling rates are always strictly positive and strictly less than unity. In the model, the restriction $f_t, q_t \in (0, 1)$ implies $v_t > 0$ and hence $\lambda_{v,t} = 0$. Assuming shocks $\{a_t\}$ are such that this restriction holds, we can invert (11) to obtain the equilibrium

[5] An alternative assumption about wages would be to follow Jung and Kuester (2011) and Freund and Rendahl (2020) and use the ad-hoc linear wage rule $w_t = \eta a_t + (1 - \eta)b$. Our qualitative results are unaffected by this choice.

[6] Den Haan et al. (2021) independently developed a similar solution to shed light on the effects of volatility shocks.

stochastic process for the job filling rate,

$$q(a_t) = \kappa/(\delta_0 + \delta_1(a_t - \bar{a})), \tag{12}$$

which is decreasing and convex in productivity. Remarkably, this result does not depend on the matching function, and instead follows directly from firms' optimal vacancy creation and the definition of marginal cost. Intuitively, higher productivity increases vacancy creation, which reduces the probability of filling any given vacancy. Convexity in the job filling rate arises because as productivity increases, the probability declines at a slower rate since it is bounded below by zero.

In equilibrium, the job filling rate is determined by labor market tightness. To derive tightness as a function of productivity, it is convenient to define the auxiliary function $\mu_q(\theta) = \mathcal{M}(1,\theta)/\theta$, which is strictly decreasing in θ and therefore invertible. Recalling that $q_t = \mathcal{M}(1,\theta_t)/\theta_t$, we can implicitly define the equilibrium tightness function as $\mu_q(\theta(a_t)) = q(a_t)$. Differentiation implies

$$\theta'(a_t) = q'(a_t)/\mu_q'(\theta(a_t)) > 0. \tag{13}$$

Since $q'(a_t), \mu_q'(\theta(a_t)) < 0$, (13) confirms that labor market tightness is increasing in productivity.

Given the equilibrium tightness function, we can use the definitions from Section 2 to define

$$\epsilon_t = \frac{\mathcal{M}_v(1,\theta(a_t))\theta(a_t)}{\mathcal{M}(1,\theta(a_t))}, \quad \sigma_t = \frac{d\ln\theta(a_t)}{d\ln(\mathcal{M}_u(1,\theta(a_t))/\mathcal{M}_v(1,\theta(a_t)))},$$

where ϵ_t is the matching elasticity and σ_t is the elasticity of substitution between job seekers and vacancies. These definitions help us uncover the nonlinearity in equilibrium tightness from $\theta''(a_t)$.

Proposition 4. *Labor market tightness, $\theta(a_t)$, is convex at a_t when $\sigma_t > 1/2$, linear at a_t when $\sigma_t = 1/2$, and concave at a_t when $\sigma_t < 1/2$.*

To interpret these conditions, it is useful to write the slope of the tightness function as

$$\theta'(a_t) = \frac{\delta_1}{\kappa}\frac{\mathcal{M}(1,\theta(a_t))}{1-\epsilon_t},$$

which shows that productivity affects $\theta'(a_t)$ through two channels. First, higher productivity generates more matches, which raises $\mathcal{M}(1,\theta(a_t))$ and $\theta'(a_t)$. Second, higher productivity affects the matching elasticity. Given Proposition 2, an increase in productivity lowers the matching elasticity and $\theta'(a_t)$ when $\sigma_t < 1$. If $\sigma_t < 1/2$, this effect dominates the first channel, so tightness is concave in productivity. When $\sigma_t = 1/2$, the two channels exactly offset, so tightness is linear in productivity. Finally, when $\sigma_t > 1/2$, the first channel dominates, so tightness is convex in productivity.

Given the equilibrium dynamics of tightness, we can use the matching function to derive the dynamics of the job finding rate. Formally, $f(a_t) = \mathcal{M}(1,\theta(a_t))$, so it is immediate that the job finding rate is increasing in productivity. As with tightness, we analyze its nonlinearity through $f''(a_t)$.

Proposition 5. *The job finding rate, $f_t = f(a_t)$, is convex at a_t when $\sigma_t > 1/(2\epsilon_t)$, linear at a_t when $\sigma_t = 1/(2\epsilon_t)$, and concave at a_t when $\sigma_t < 1/(2\epsilon_t)$.*

To interpret these conditions, it is useful to write the slope of the job finding rate function as

$$f'(a_t) = \mathcal{M}_v(1, \theta(a_t))\theta'(a_t),$$

which shows that productivity also affects $f'(a_t)$ through two competing channels. First, higher productivity raises labor market tightness, which lowers its marginal product, $\mathcal{M}_v(1, \theta(a_t))$, due to diminishing returns to vacancy creation. This generates concavity in the job finding rate. Second, higher productivity affects the responsiveness of tightness itself through $\theta'(a_t)$. As Proposition 4 shows, this effect is positive when $\sigma_t > 1/2$, generating convexity. If $\sigma_t > 1/(2\epsilon_t) > 1/2$, it is strong enough to dominate the first channel, making the job finding rate convex in productivity. When $\sigma_t = 1/(2\epsilon_t)$, the two channels exactly offset, so the finding rate is linear in productivity. Finally, when $\sigma_t < 1/(2\epsilon_t)$, the first channel dominates, so the finding rate is concave in productivity.

4.2 EXAMPLES

We can generate additional insights by considering specific matching functions.

CES Matching Function Recall the CES matching function,

$$\mathcal{M}(u_{t-1}, v_t) = \begin{cases} \phi \left(\vartheta u_{t-1}^{(\sigma-1)/\sigma} + (1-\vartheta) v_t^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} & \sigma \neq 1, \\ \phi u_{t-1}^{\vartheta} v_t^{1-\vartheta} & \sigma = 1, \end{cases} \tag{14}$$

where $\phi > 0$ is matching efficiency and $\vartheta \in (0, 1)$ governs the importance of unemployment. In this case, the elasticity of substitution, σ, is fixed, while the matching elasticity takes the specific form $\epsilon_t = (1-\vartheta)(\phi/q(a_t))^{(\sigma-1)/\sigma}$. In line with Corollary 1, the matching elasticity is procyclical when $\sigma > 1$, constant when $\sigma = 1$, and countercyclical when $\sigma < 1$. Using these properties, we can derive sufficient conditions for global convexity or concavity of the job finding rate function.

Corollary 2. *Suppose $\mathcal{M}(u_{t-1}, v_t)$ satisfies (14). Then $\sigma > \frac{1}{2(1-\vartheta)\phi^{(\sigma-1)/\sigma}} \geq 1$ implies that $f(a_t)$ is globally convex, $\sigma < \frac{1}{2(1-\vartheta)\phi^{(\sigma-1)/\sigma}} \leq 1$ implies that $f(a_t)$ is globally concave, and $\sigma = \frac{1}{2(1-\vartheta)} = 1$ implies that $f(a_t)$ is globally linear.*

DRW Matching Function A small but influential set of papers (e.g., Hagedorn and Manovskii, 2008; Petrosky-Nadeau et al., 2018) use the function introduced by Den Haan et al. (2000, DRW):

$$\mathcal{M}(u_{t-1}, v_t) = u_{t-1} v_t / (u_{t-1}^{\iota} + v_t^{\iota})^{1/\iota}. \tag{15}$$

In this case, $\iota > 0$ and the elasticity of substitution is fixed at $1/(1+\iota) < 1$. The matching elasticity satisfies $\epsilon_t = q(a_t)^{\iota}$ and is always countercyclical according to Proposition 2. Thus, this specification is inconsistent with the empirical estimates in Lange and Papageorgiou (2020). While it is often justified by appealing to the fact that it guarantees bounded job finding and filling rates without the feasibility condition (2), it also has significant effects on nonlinear labor market dynamics.

Corollary 3. *Suppose $\mathcal{M}(u_{t-1}, v_t)$ satisfies (15). Then $\iota > 1$ implies $f(a_t)$ is globally concave.*

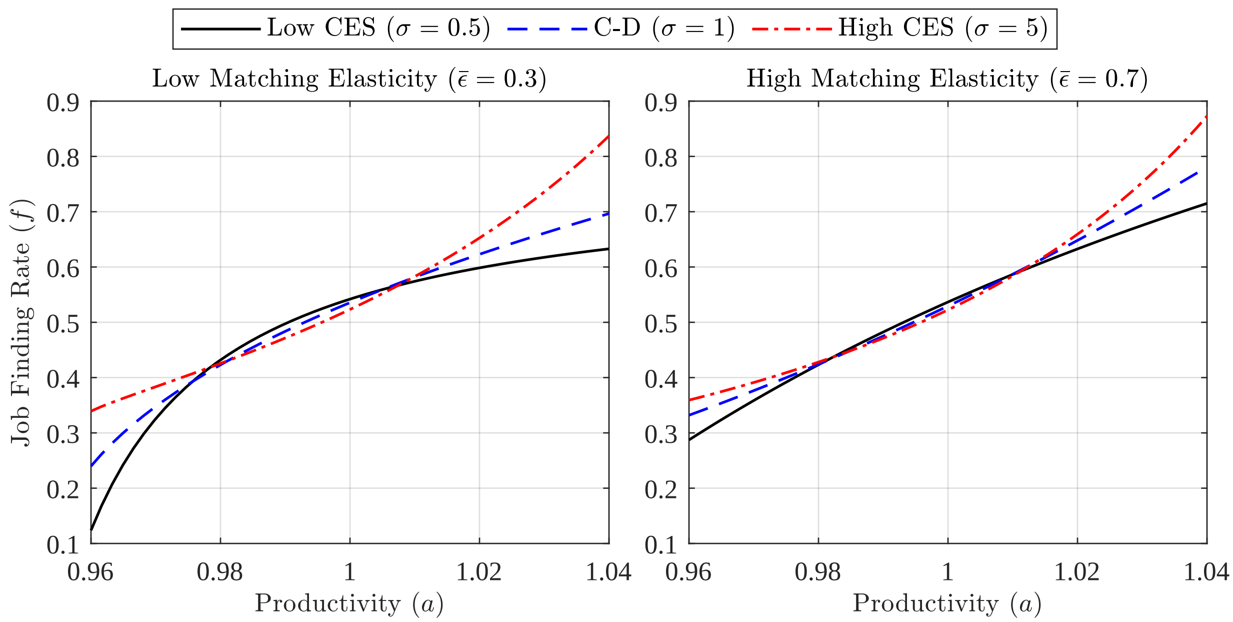

Figure 1: Nonlinearity of the job finding rate function.

4.3 ILLUSTRATION To see the nonlinearity embedded in the job finding rate function, Figure 1 plots $f(a_t)$ using a CES matching function with $\sigma \in \{0.5, 1, 5\}$ and $\bar{\epsilon} \in \{0.3, 0.7\}$. Our choice of $\sigma = 0.5$ corresponds to $\iota = 1$ under a DRW matching function, which is comparable to values used in the literature (e.g., Petrosky-Nadeau et al. (2018) set $\iota = 1.25$, which would produce even more concavity than $\iota = 1$), while $\sigma = 1$ is the Cobb-Douglas case and $\sigma = 5$ generates a procyclical matching elasticity in line with Lange and Papageorgiou (2020). Our choices for $\bar{\epsilon}$ captures the range of estimates in the empirical literature. All other parameters are set using the strategy in Section 5, which ensures the mean unemployment rate is fixed across the different specifications.

Following Proposition 5, the nonlinearity around steady state depends on whether $\sigma \lesseqgtr 1/(2\bar{\epsilon})$. When $\bar{\epsilon} = 0.3$, the threshold for convexity is relatively high, so the job finding rate is concave in the Cobb-Douglas case and features pronounced concavity when $\sigma = 0.5$. When $\bar{\epsilon} = 0.7$, the threshold is lower, which results in far weaker concavity when $\sigma = 0.5$ and mild convexity in the Cobb-Douglas case. When $\sigma = 5$, there is pronounced convexity for both values of $\bar{\epsilon}$. These results illustrate the importance of the matching function parameters for nonlinear labor market dynamics.

4.4 NONLINEAR UNEMPLOYMENT DYNAMICS Since the matching function affects job finding rate dynamics, it also affects unemployment dynamics via its law of motion. Differentiating (6) yields $\partial u_t/\partial a_t = -u_{t-1}f'(a_t)$, which shows that the size of the unemployment response to a change in productivity is larger when unemployment is already elevated and when the job finding rate function is steeper. Intuitively, unemployment responds more when a change in the finding rate is applied to a larger pool of workers or when the finding rate itself changes by a larger amount.

To understand whether the matching function amplifies or dampens nonlinear unemployment dynamics, first note that u_{t-1} and a_t are negatively correlated because higher unemployment is driven by low productivity shocks. Therefore, u_{t-1} and $f'(a_t)$ are positively correlated when the job finding rate function is concave. Hence, a concave job finding rate function amplifies nonlinear unemployment dynamics since periods of high unemployment coincide with larger finding rate responses to productivity shocks. In contrast, a convex job finding function dampens the nonlinearity of unemployment because high unemployment tends to occur with smaller finding rate responses.

5 Quantitative Results

Our analysis highlights the importance of the matching elasticity and elasticity of substitution. To transparently quantify the mechanism, we use a CES functional form and report results for the same values of the elasticity of substitution, σ, and steady-state matching elasticity, $\bar{\epsilon}$, used in Section 4.3.

Each period in the model denotes 1 month, so the discount factor, β, is set to 0.9983, which corresponds to an average annual real interest rate of 2%. The coefficient of relative risk aversion, γ, is set to 1, consistent with log utility. The remaining parameters are based on U.S. data from 1955 to 2019. The steady-state job separation rate, $\bar{s}$, is set to its sample mean 0.0326, which we compute following Shimer (2012). The persistence ($\rho_a = 0.8826$) and standard deviation ($\sigma_a = 0.0062$) of productivity are set to match the autocorrelation and standard deviation of detrended productivity.

To isolate the impact of the matching function on higher-order labor market dynamics, we hold the mean and standard deviation of the unemployment rate fixed across $(\sigma, \bar{\epsilon})$ pairs. In particular, under each specification we estimate the vacancy posting cost, κ, and flow value of unemployment, b, to target the mean unemployment rate and the standard deviation of the detrended unemployment rate in our data sample. In addition, we estimate the bargaining power parameter, η, to target the wage-productivity elasticity.[7] Each specification is able to perfectly match these empirical targets.

We set the steady-state job filling rate to 0.3306, which corresponds to a quarterly filling rate of 0.7 (Den Haan et al., 2000). The steady-state job finding rate is endogenously pinned down by the mean unemployment rate since $\bar{f} = \bar{s}(1-\bar{u})/\bar{u}$, and $\bar{u}$ is determined by the vacancy posting cost,

[7] The empirical targets are based on quarterly data. Each period in the model denotes 1 month, so we aggregate the simulated time series to a quarterly frequency to match the frequency of labor productivity in the data. To facilitate comparison with the literature, we detrend actual data using a Hodrick and Prescott (1997) filter with a smoothing parameter of $1{,}600$. We detrend simulated data by computing percent deviations from the short-sample time averages. The wage rate (w_t) is defined as the product of the labor share and labor productivity (a_t) in the nonfarm business sector (Hagedorn and Manovskii, 2008). The wage elasticity is the slope coefficient from regressing w_t on an intercept and a_t.

κ. Given $\bar{q}$, $\bar{f}$, and a $(\sigma, \bar{\epsilon})$ pair, we pin down ϑ and ϕ using the following steady-state restrictions:

$$\phi = \begin{cases} \left[\bar{\epsilon}\bar{q}^{(\sigma-1)/\sigma} + (1-\bar{\epsilon})\bar{f}^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)} & \sigma \neq 1, \\ \bar{q}^{\bar{\epsilon}}\bar{f}^{1-\bar{\epsilon}} & \sigma = 1, \end{cases}$$

$$\vartheta = (1-\bar{\epsilon})(\bar{f}/\phi)^{(\sigma-1)/\sigma}.$$

This ensures each matching function has similar first-order properties, in line with Proposition 1.

Solution Method To quantify the nonlinearities, we solve the model globally using the policy function iteration algorithm in Richter et al. (2014), which is based on the theoretical work in Coleman (1991). The algorithm minimizes the Euler equation errors on each node in the state space and computes the maximum change in the policy functions. It then iterates until the maximum change is below a specified tolerance criterion. Appendix B describes the solution method in more detail.

Estimates Table 2a reports the estimated parameters, (κ, b, η), and implied matching function parameters, (ϕ, ϑ), given the steady-state matching elasticity, $\bar{\epsilon}$, and the elasticity of substitution, σ.[8] All of the parameter estimates are in line with values that are commonly used in the literature.

Higher-Order Moments Table 2b shows key untargeted moments across the $(\sigma, \bar{\epsilon})$ pairs. Consider first the specifications where $\bar{\epsilon} = 0.3$, which is close to recent estimates of the mean matching elasticity reported in Table 1. When $\sigma = 0.5$ and the matching elasticity is countercyclical, positive productivity shocks are dampened relative to negative shocks. As a result, job finding rate dynamics exhibit significant negative skewness (-1.4), which amplifies the positive skewness and kurtosis of the unemployment rate (2.37 and 9.78). These outcomes are flipped when $\sigma = 5$ and the matching elasticity is procyclical. The job finding rate becomes positively skewed (0.33), and the positive skewness and kurtosis of the unemployment rate are considerably weaker (0.29 and 0.04).

Qualitatively similar patterns emerge when $\bar{\epsilon} = 0.7$, though the differences across σ values are much less pronounced. In line with the logic from Proposition 5, a higher mean matching elasticity lowers the threshold that σ must exceed for the job finding rate function to be convex in productivity. Therefore, there is much less negative skewness when $\sigma = 0.5$ (-0.29), which results in less amplification of the positive skewness and kurtosis of the unemployment rate (0.95 and 1.55). When $\sigma = 5$, the job finding rate is even more positively skewed than when $\bar{\epsilon} = 0.3$ (0.49), which results in almost no skewness or kurtosis in the unemployment rate (0.15 and -0.08).

Crucially, the large variation in higher-order labor market moments is driven by modest cyclical movements in the matching elasticity. When $\sigma \neq 1$, the standard deviation of the matching elasticity ranges from 0.03 to 0.07. This modest variation implies that the matching elasticity would

[8] Consistent with Hagedorn and Manovskii (2008), the baseline model requires a b that is close to the marginal product of labor in order to generate realistic labor market volatility. Appendix C shows that if we introduce home production, we can set $b = 0.4$ so it resembles an unemployment benefit while achieving the same labor market volatility.

$\bar{\epsilon}$	0.3			0.7		
σ	0.5	1.0	5.0	0.5	1.0	5.0
Vacancy Posting Cost (κ)	0.0794	0.0610	0.0507	0.3848	0.3493	0.3367
Flow Value of Unemployment (b)	0.9716	0.9777	0.9815	0.9243	0.9302	0.9328
Worker Bargaining Power (η)	0.1276	0.1320	0.1327	0.0515	0.0534	0.0535
Matching Efficiency (ϕ)	0.4540	0.4645	0.4683	0.3733	0.3811	0.3879
Unemployment Weight (ϑ)	0.5880	0.7000	0.7729	0.2096	0.3000	0.3841

(a) Estimated and implied parameter values.

$\bar{\epsilon}$	0.3			0.7		
σ	0.5	1.0	5.0	0.5	1.0	5.0
$Skew(f)$	−1.40	−0.58	0.33	−0.29	0.12	0.49
$Skew(u)$	2.37	1.35	0.29	0.95	0.49	0.15
$Kurt(f)$	3.35	0.75	0.15	0.08	−0.06	0.32
$Kurt(u)$	9.78	3.63	0.04	1.55	0.33	−0.08
$SD(\epsilon)$	0.07	0.00	0.07	0.04	0.00	0.03
$Corr(\epsilon, u)$	0.96	0.00	−0.98	0.97	0.00	−0.98

(b) Higher-order moments. All specifications generate the same $E(u)$, $SD(u)$, and $Slope(w, a)$.

Table 2: Quantitative results.

rarely leave the range of estimates in Table 1, given a mean in that range. It also aligns with the direct evidence of cyclical variation provided by Lange and Papageorgiou (2020). They find the matching elasticity is procyclical, varying between 0.15 and 0.30 with a standard deviation of 0.04. These estimates imply far less nonlinearity in labor market dynamics than the literature has recently emphasized (e.g., Ferraro, 2018; Petrosky-Nadeau and Zhang, 2017; Petrosky-Nadeau et al., 2018).

The dependence of the higher-order moments on the elasticity of substitution suggests we could identify σ by adding them as empirical targets. We explored this strategy but did not find it compelling for two reasons. First, the estimate for σ was sensitive to the targeted higher-order moments (e.g., $Skew(u)$ or $Skew(f)$) and the steady-state matching elasticity $\bar{\epsilon}$. Second, identifying σ using higher-order moments assumes that cyclical variation in the matching elasticity is the only driver of nonlinear dynamics. This contradicts existing work such as Dupraz et al. (2019), who show how downward wage rigidity can also create nonlinear labor market dynamics. Thus, the estimate of σ would also be sensitive to the inclusion of model ingredients that affect higher-order moments. The macro implications of the matching function, instead, motivate additional microeconometric work.

Impulse Responses A growing literature uses the search and matching model as a lens for understanding deep recessions and business cycle asymmetries (e.g., Dupraz et al., 2019; Petrosky-Nadeau and Zhang, 2017; Petrosky-Nadeau et al., 2018). Our analysis shows the matching function specification plays a crucial role in this setting. While the skewness and kurtosis moments capture

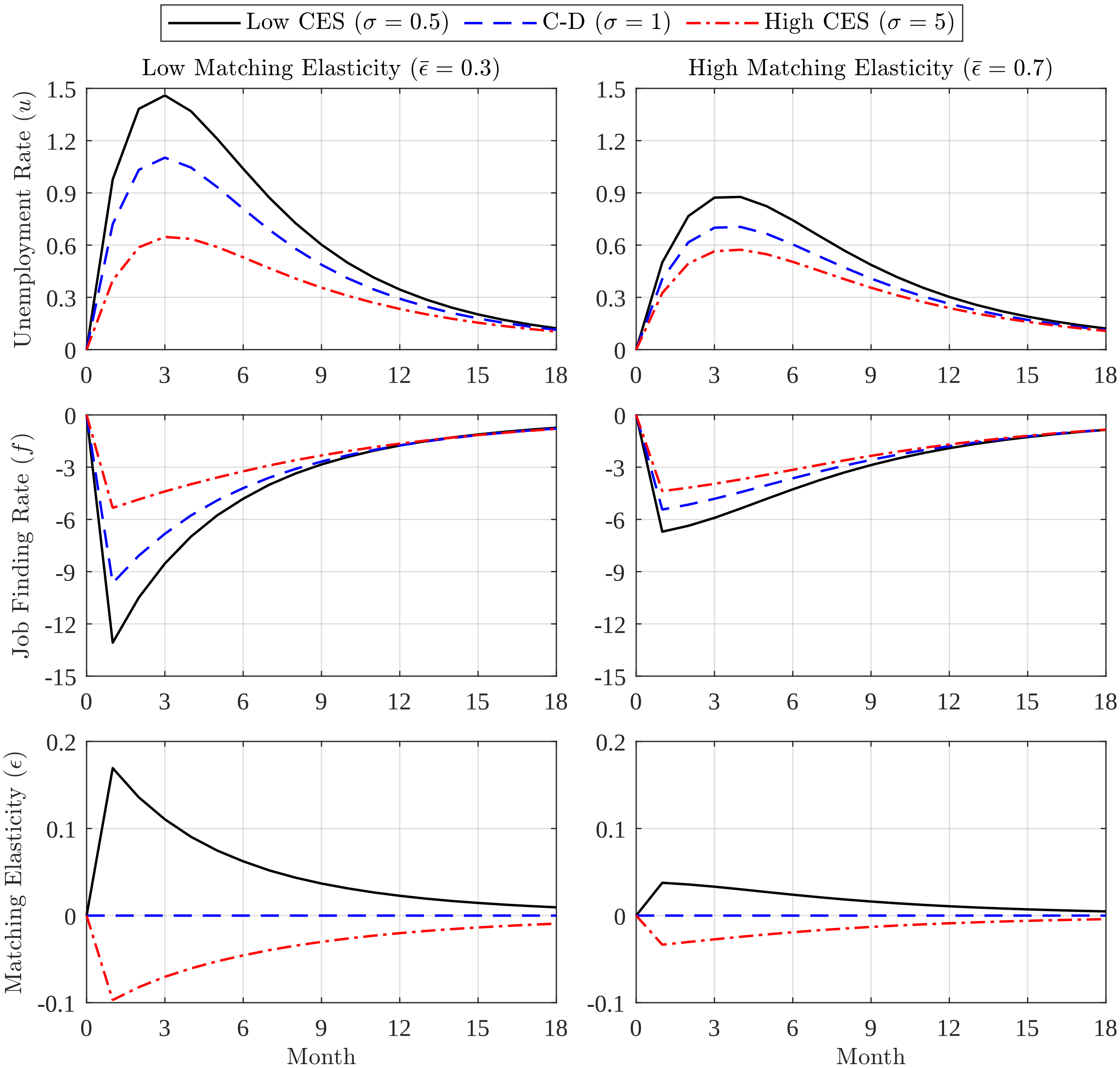

Figure 2: Generalized impulse responses to a -2 SD shock initialized in a recession ($u_0 = 7.5\%$). The job finding and unemployment rates are percentage point changes and the matching elasticity is a level change.

some of this effect, Figure 2 provides further context by plotting generalized impulse responses of the unemployment and job finding rates to a 2 standard deviation negative productivity shock.[9] We allow for state-dependence by initializing the simulations in a recession ($u_0 = 7.5\%$). When we alternatively initialize the simulations at steady state ($u_0 = 5.9\%$), the responses are similar across matching function specifications. This intuitively follows from the fact that our parameter calibration strategy ensures that all matching function specifications generate similar first-order dynamics.

[9]Following Koop et al. (1996), the response of x_{t+h} over horizon h is given by $\mathcal{G}_t(x_{t+h}|\varepsilon_{a,t+1} = -2, \mathbf{z}_t) = E_t[x_{t+h}|\varepsilon_{a,t+1} = -2, \mathbf{z}_t] - E_t[x_{t+h}|\mathbf{z}_t]$, where $\mathbf{z}_t$ is a vector of initial states and -2 is the shock size in period $t+1$.

Large differences in the impulse responses emerge when the shock hits in a recessionary state and the mean matching elasticity is low ($\bar{\epsilon} = 0.3$). When $\sigma = 0.5$ and the job finding rate is a concave function of productivity, the matching function generates an unemployment rate response that is more than double the response when $\sigma = 5$. The larger response is driven by a larger decline in the job finding rate, which follows from the countercyclical increase in the matching elasticity. If the mean matching elasticity is higher ($\bar{\epsilon} = 0.7$), the differences in the responses across σ are still apparent, but not as pronounced. This again shows that the average level and cyclicality of the matching elasticity are important to account for when studying nonlinear business cycle dynamics.

6 NORMATIVE IMPLICATIONS

This section shows the cyclicality of the matching function has normative implications, which affect the wedges that restore efficiency and the response of the efficient real interest rate to shocks.

6.1 EFFICIENT FISCAL POLICY The equilibrium of a search and matching model is generally inefficient due to two externalities in the matching process (Hosios, 1990). First, when a firm posts a new vacancy, it imposes a positive externality on unemployed workers who face a higher job finding rate. Second, the same vacancy posting imposes a negative externality on other firms who face lower job filling rates and a higher marginal cost of vacancy creation today and in the future.

To see how the matching function affects these externalities and the efficient policy responses, we compare the equilibrium to the solution of a planning problem in which both externalities are internalized. The problem and solution are described in Appendix A. The key optimality condition is

$$\frac{\kappa - \lambda_{v,t}}{\mathcal{M}_v(u_{t-1}, v_t)} = a_t - b + E_t \left[x_{t+1} \frac{\kappa - \lambda_{v,t+1}}{\mathcal{M}_v(u_t, v_{t+1})} (1 - \bar{s} - \mathcal{M}_u(u_t, v_{t+1})) \right], \tag{16}$$

which determines the optimal level of vacancies by setting the social marginal cost (SMC) of a vacancy to its social marginal benefit (SMB). The gaps between the SMC and SMB and the private marginal cost (PMC) and private marginal benefit (PMB) reflect inefficiencies of the equilibrium.

To characterize these gaps, we follow the public finance literature and solve for the wedges—state-dependent, linear taxes—that equate the two solutions. Let $\tau_{v,t}$ denote a tax on vacancy creation, v_t, and $\tau_{n,t}$ a tax on a firm's payroll, n_{t-1}, so that the firm's flow profits are given by $(a_t - w_t)n_t - (1 + \tau_{v,t})\kappa v_t - \tau_{n,t} n_{t-1}$.[10] Then the firm's optimal vacancy creation choice is given by

$$\frac{\kappa - \lambda_{v,t}}{q_t} = \frac{1-\eta}{1+\tau_{v,t}}(a_t - b) + E_t \left[\tilde{x}_{t+1} \frac{\kappa - \lambda_{v,t+1}}{q_{t+1}} \left(1 - \bar{s} - \frac{1}{1+\tau_{v,t+1}} \frac{q_{t+1}}{\kappa - \lambda_{v,t+1}} (\kappa \eta \theta_{t+1} + \tau_{n,t+1}) \right) \right],$$

where $\tilde{x}_{t+1} \equiv x_{t+1}(1 + \tau_{v,t+1})/(1 + \tau_{v,t})$. We can now solve for the wedges that restore efficiency.

[10] Placing a wedge on n_t would be equivalent. We put the wedge on n_{t-1} since it is easier to compute and interpret.

Proposition 6. *The efficiency-restoring wedges are given by*

$$\tau_v(\theta_t) = (1-\eta)/\epsilon(\theta_t) - 1,$$
$$\tau_n(\theta_t) = \theta_t((\kappa - \lambda_{v,t})\tau_v(\theta_t) - \eta\lambda_{v,t}),$$

where each wedge is evaluated at the solution to the planning problem. Furthermore, $\tau_v'(\theta_t) > 0$ *when* $\sigma_t < 1$*, and* $\tau_n'(\theta_t) > 0$ *when* $\sigma_t < \frac{1-\eta}{\eta}\frac{1-\epsilon_t}{\epsilon_t}$ *for all* $\theta_t > 0$*.*

The expression for $\tau_{v,t}$ shows how the vacancy tax balances the externalities. Note that $1-\eta$ is the ratio of the period-t PMB, $(1-\eta)(a_t - b)$, to the period-t SMB, $a_t - b$. The matching elasticity $\epsilon_t = \frac{\kappa/q_t}{\kappa/\mathcal{M}_v(u_{t-1},v_t)}$ is the ratio of the PMC to the SMC. The sign of the wedge depends on which ratio is larger. For example, $\tau_{v,t} > 0$ when $\epsilon_t < 1-\eta$ and the marginal cost gap is smaller than the marginal benefit gap. In this case, there is inefficiently high private vacancy creation and the negative externality on firms dominates the positive externality on workers. A positive vacancy wedge dampens the incentive for private vacancy creation, restoring efficiency of the equilibrium.

Crucially, $\tau_{v,t}$ co-moves negatively with the matching elasticity, indicating its time-varying strength. For example, if the matching function is CES, then the matching elasticity is countercyclical and $\tau_{v,t}$ is procyclical when $\sigma < 1$ because the gap between private and efficient vacancy creation is larger in booms. In contrast, $\tau_{v,t}$ is countercyclical when $\sigma > 1$ because the gap is larger in recessions. Finally, in the knife-edge case where $\sigma = 1$, the efficient vacancy tax is constant. Thus, the matching function specification is crucial for implementing efficient taxes on vacancy creation.

The payroll tax ($\tau_{n,t} > 0$) accounts for the gap between the period-$t+1$ SMB and PMB. Intuitively, private vacancy creation boosts employment today, which lowers u_t and raises the marginal cost of vacancy creation in the future. A payroll tax is necessary to limit private vacancy creation in period t, undoing the negative externality. Restricting attention to $\theta_t > 0$ so that $\lambda_{v,t} = 0$ and $\tau_{n,t} = \kappa\theta_t\tau_{v,t}$, its time-variation is determined by two forces. The first is procyclical variation in tightness. The second is variation in $\tau_{v,t}$, which is decreasing in tightness when $\sigma_t > 1$. However, as long as $\sigma_t < \frac{1-\eta}{\eta}\frac{1-\epsilon_t}{\epsilon_t}$, this force is dominated by or amplifies the first channel so that $\tau_{n,t}$ is procyclical. Understanding the true matching function is again vital for implementing the efficient tax.

6.2 OPTIMAL MONETARY POLICY When the real allocation is efficient, the corresponding real interest rate, r_t^*, serves as the key target for monetary policy in the presence of nominal rigidities.[11] To understand how the nonlinearities in the matching function impact the optimal monetary policy response to productivity shocks, Figure 3 plots generalized impulse responses of r_t^* to a 2 standard

[11] The optimality of targeting r_t^* requires appropriate fiscal policies to correct for the matching externalities described above and for the inefficient markups created by price-setting power. See Lepetit (2020) for an example of optimal monetary policy without fiscal policies in a search and matching model with the Cobb-Douglas matching function.

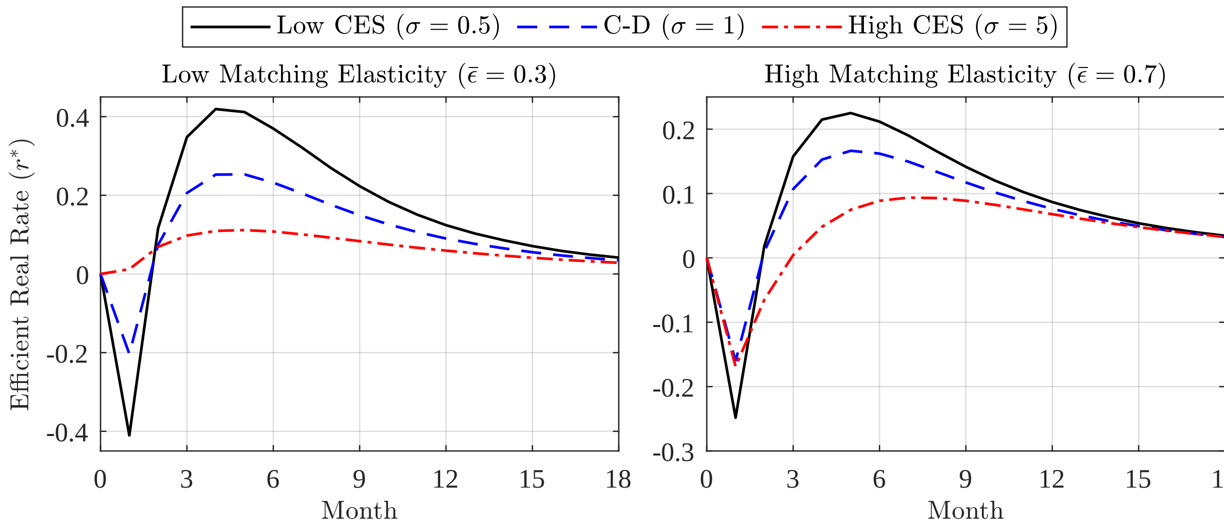

Figure 3: Percentage point responses to a -2 SD shock initialized in a recession ($u_0 = 7.5\%$).

deviation negative labor productivity shock when the economy begins in a recession ($u_0 = 7.5\%$).[12]

The responses of r_t^* are driven by expected changes in consumption growth. Since the consumption response largely follows the negative of the unemployment rate response in Figure 2, r_t^* inherits its nonlinear dynamics, which are affected by the matching function specification. Consider the responses when $\bar{\epsilon} = 0.3$. When $\sigma = 0.5$, the higher peak unemployment response leads to a larger decline in consumption and a more volatile r_t^* response than when $\sigma = 5$. The initial decline in r_t^* occurs because consumption growth first declines in response to the shock, before increasing as the shock dissipates. This effect disappears when $\sigma = 5$ due to the weaker unemployment response. Similar results emerge when $\bar{\epsilon} = 0.7$, except the differences in the r^* responses are muted with less curvature in the matching function. Just like the optimal wedges, these results show the importance of knowing the matching function for the conduct of optimal monetary policy.

7 CONCLUSION

The Cobb-Douglas matching function is ubiquitous in search and matching models, even though it imposes a constant elasticity of matches with respect to vacancies that is unlikely to hold empirically. To examine the implications of this discrepancy, we use a general constant-returns-to-scale matching function to derive conditions that determine how the cyclicality of the matching elasticity amplifies or dampens the nonlinear dynamics of the job finding and unemployment rates. We then show these effects are quantitatively large and driven by modest variation in the matching elasticity.

While richer models could affect the strength of the nonlinearities, the Cobb-Douglas match-

[12]Following the approach in Section 5, we set the vacancy posting cost, κ, and flow value of unemployment, b, in the efficient equilibrium so that the mean and standard deviation of the unemployment rate are fixed across $(\sigma, \bar{\epsilon})$ pairs.

ing function is not without loss of generality. The cyclicality of the matching elasticity that ensues when deviating from Cobb-Douglas would feed into to job finding and unemployment rate dynamics in any search and matching model, so it is important for future research to show how alternative matching functions affect their results. Furthermore, we hope our analysis motivates empirical work that provides additional clarity on the true nature of the matching frictions in the labor market.

REFERENCES

ABBRITTI, M. AND S. FAHR (2013): "Downward Wage Rigidity and Business Cycle Asymmetries," *Journal of Monetary Economics*, 60, 871–886, https://doi.org/10.1016/j.jmoneco.2013.08.001.

ANDOLFATTO, D. (1996): "Business Cycles and Labor-Market Search," *American Economic Review*, 86, 112–132.

ARSENEAU, D. M. AND S. K. CHUGH (2012): "Tax Smoothing in Frictional Labor Markets," *Journal of Political Economy*, 120, 926–985, https://doi.org/10.1086/668837.

BARNICHON, R. AND A. FIGURA (2015): "Labor Market Heterogeneity and the Aggregate Matching Function," *American Economic Journal: Macroeconomics*, 7, 222–249, https://doi.org/10.1257/mac.20140116.

BENHABIB, J., R. ROGERSON, AND R. WRIGHT (1991): "Homework in Macroeconomics: Household Production and Aggregate Fluctuations," *Journal of Political Economy*, 99, 1166–1187, https://doi.org/10.1086/261796.

BERNSTEIN, J., A. W. RICHTER, AND N. A. THROCKMORTON (2021): "Cyclical Net Entry and Exit," *European Economic Review*, 136, https://doi.org/10.1016/j.euroecorev.2021.103752.

BLANCHARD, O. J. AND P. DIAMOND (1989): "The Beveridge Curve," *Brookings Papers on Economic Activity*.

BLEAKLEY, H. AND J. C. FUHRER (1997): "Shifts in the Beveridge Curve, job matching, and labor market dynamics," *New England Economic Review*, 3–19.

BOROWCZYK-MARTINS, D., G. JOLIVET, AND F. POSTEL-VINAY (2013): "Accounting for endogeneity in matching function estimation," *Review of Economic Dynamics*, 16, 440–451, https://doi.org/10.1016/j.red.2012.07.001.

COLEMAN, II, W. J. (1991): "Equilibrium in a Production Economy with an Income Tax," *Econometrica*, 59, 1091–1104, https://doi.org/10.2307/2938175.

ŞAHIN, A., J. SONG, G. TOPA, AND G. L. VIOLANTE (2014): "Mismatch Unemployment," *American Economic Review*, 104, 3529–3564, https://doi.org/10.1257/aer.104.11.3529.

DEN HAAN, W., L. B. FREUND, AND P. RENDAHL (2021): "Volatile Hiring: Uncertainty in Search and Matching Models," *Journal of Monetary Economics*, 123, 1–18, https://doi.org/10.1016/j.jmoneco.2021.07.008.

DEN HAAN, W. J., G. RAMEY, AND J. WATSON (2000): "Job Destruction and Propagation of Shocks," *American Economic Review*, 90, 482–498, https://doi.org/10.1257/aer.90.3.482.

DUPRAZ, S., E. NAKAMURA, AND J. STEINSSON (2019): "A Plucking Model of Business Cycles," NBER Working Paper 26351, https://doi.org/10.3386/w26351.

FAIA, E. (2009): "Ramsey monetary policy with labor market frictions," *Journal of Monetary Economics*, 56, 570–581, https://doi.org/10.1016/j.jmoneco.2009.03.009.

FERRARO, D. (2018): "The Asymmetric Cyclical Behavior of the U.S. Labor Market," *Review of Economic Dynamics*, 30, 145–162, https://doi.org/10.1016/j.red.2018.05.005.

FERRARO, D. AND G. FIORI (2021): "Non-linear employment effects of tax policy," International Finance Discussion Paper 1333, https://ideas.repec.org/p/fip/fedgif/1333.html.

FREUND, L. AND P. RENDAHL (2020): "Unexpected Effects: Uncertainty, Unemployment, and Inflation," CEPR Discussion Paper 14690.

GARCIA, C. B. AND W. I. ZANGWILL (1981): *Pathways to Solutions, Fixed Points and Equilibria*, Prentice-Hall series in computational mathematics, Prentice-Hall.

HAGEDORN, M. AND I. MANOVSKII (2008): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies Revisited," *American Economic Review*, 98, 1692–1706, https://doi.org/10.1257/aer.98.4.1692.

HAIRAULT, J.-O., F. LANGOT, AND S. OSOTIMEHIN (2010): "Matching frictions, unemployment dynamics and the cost of business cycles," *Review of Economic Dynamics*, 13, 759–779, https://doi.org/10.1016/j.red.2010.05.001.

HALL, R. E. (2005): "Employment Fluctuations with Equilibrium Wage Stickiness," *American Economic Review*, 95, 50–65, https://doi.org/10.1257/0002828053828482.

HALL, R. E. AND P. R. MILGROM (2008): "The Limited Influence of Unemployment on the Wage Bargain," *American Economic Review*, 98, 1653–1674, https://doi.org/10.1257/aer.98.4.1653.

HALL, R. E. AND S. SCHULHOFER-WOHL (2018): "Measuring Job-Finding Rates and Matching Efficiency with Heterogeneous Job-Seekers," *American Economic Journal: Macroeconomics*, 10, 1–32, https://doi.org/10.1257/mac.20170061.

HASHIMZADE, N. AND S. ORTIGUEIRA (2005): "Endogenous Business Cycles With Frictional Labour Markets," *Economic Journal*, 115, 161–175, https://doi.org/10.1111/j.0013-0133.2005.00985.x.

HODRICK, R. J. AND E. C. PRESCOTT (1997): "Postwar U.S. Business Cycles: An Empirical Investigation," *Journal of Money, Credit and Banking*, 29, 1–16, https://doi.org/10.2307/2953682.

HOSIOS, A. J. (1990): "On The Efficiency of Matching and Related Models of Search and Unemployment," *Review of Economic Studies*, 57, 279–298, https://doi.org/10.2307/2297382.

JUNG, P. AND K. KUESTER (2011): "The (un)importance of unemployment fluctuations for the welfare cost of business cycles," *Journal of Economic Dynamics and Control*, 35, 1744–1768, https://doi.org/10.1016/j.jedc.2011.05.008.

——— (2015): "Optimal Labor-Market Policy in Recessions," *American Economic Journal: Macroeconomics*, 7, 124–156, https://doi.org/10.1257/mac.20130028.

KOOP, G., M. H. PESARAN, AND S. M. POTTER (1996): "Impulse Response Analysis in Nonlinear Multivariate Models," *Journal of Econometrics*, 74, 119–147, https://doi.org/10.1016/0304-4076(95)01753-4.

KOPECKY, K. AND R. SUEN (2010): "Finite State Markov-Chain Approximations to Highly Persistent Processes," *Review of Economic Dynamics*, 13, 701–714, https://doi.org/10.1016/j.red.2010.02.002.

LANGE, F. AND T. PAPAGEORGIOU (2020): "Beyond Cobb-Douglas: Flexibly Estimating Matching Functions with Unobserved Matching Efficiency," NBER Working Paper 26972, https://doi.org/10.3386/w26972.

LEPETIT, A. (2020): "Asymmetric Unemployment Fluctuations and Monetary Policy Trade-Offs," *Review of Economic Dynamics*, 36, 29–45, https://doi.org/10.1016/j.red.2019.07.005.

LJUNGQVIST, L. AND T. J. SARGENT (2017): "The Fundamental Surplus," *American Economic Review*, 107, 2630–2665, https://doi.org/10.1257/aer.20150233.

MERZ, M. (1995): "Search in the Labor Market and the Real Business Cycle," *Journal of Monetary Economics*, 36, 269–300, https://doi.org/10.1016/0304-3932(95)01216-8.

MICHAILLAT, P. AND E. SAEZ (2021): "Beveridgean unemployment gap," *Journal of Public Economics Plus*, 2, https://doi.org/10.1016/j.pubecp.2021.100009.

PETRONGOLO, B. AND C. A. PISSARIDES (2001): "Looking into the Black Box: A Survey of the Matching Function," *Journal of Economic Literature*, 39, 390–431, https://doi.org/10.1257/jel.39.2.390.

PETROSKY-NADEAU, N. AND E. WASMER (2017): *Labor, Credit, and Goods Markets: The Macroeconomics of Search and Unemployment*, Cambridge, MA: MIT Press, 1st ed.

PETROSKY-NADEAU, N. AND L. ZHANG (2017): "Solving the Diamond-Mortensen-Pissarides model accurately," *Quantitative Economics*, 8, 611–650, https://doi.org/10.3982/QE452.

——— (2021): "Unemployment crises," *Journal of Monetary Economics*, 117, 335–353, https://doi.org/10.1016/j.jmoneco.2020.01.009.

PETROSKY-NADEAU, N., L. ZHANG, AND L.-A. KUEHN (2018): "Endogenous Disasters," *American Economic Review*, 108, 2212–2245, https://doi.org/10.1257/aer.20130025.

PISSARIDES, C. A. (2009): "The Unemployment Volatility Puzzle: Is Wage Stickiness the Answer?" *Econometrica*, 77, 1339–1369, https://doi.org/10.3982/ECTA7562.

PIZZINELLI, C., K. THEODORIDIS, AND F. ZANETTI (2020): "State Dependence in Labor Market Fluctuations," *International Economic Review*, 61, 1027–1072, https://doi.org/10.1111/iere.12448.

RICHTER, A. W., N. A. THROCKMORTON, AND T. B. WALKER (2014): "Accuracy, Speed and Robustness of Policy Function Iteration," *Computational Economics*, 44, 445–476, https://doi.org/10.1007/s10614-013-9399-2.

ROGERSON, R. AND R. SHIMER (2011): "Search in Macroeconomic Models of the Labor Market," in *Handbook of Labor Economics*, ed. by O. Ashenfelter and D. Card, Elsevier, vol. 4, chap. 7, 1341–1393.

ROUWENHORST, K. G. (1995): "Asset Pricing Implications of Equilibrium Business Cycle Models," in *Frontiers of Business Cycle Research*, ed. by T. F. Cooley, Princeton, NJ: Princeton University Press, 294–330.

SEDLÁČEK, P. (2016): "The aggregate matching function and job search from employment and out of the labor force," *Review of Economic Dynamics*, 21, 16–28, https://doi.org/10.1016/j.red.2016.03.001.

SHIMER, R. (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 95, 25–49, https://doi.org/10.1257/0002828053828572.

——— (2012): "Reassessing the Ins and Outs of Unemployment," *Review of Economic Dynamics*, 15, 127–148, https://doi.org/10.1016/j.red.2012.02.001.

SIMS, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20, 1–20, https://doi.org/10.1023/A:1020517101123.

STEVENS, M. (2007): "New Microfoundations For The Aggregate Matching Function," *International Economic Review*, 48, 847–868, https://doi.org/10.1111/j.1468-2354.2007.00447.x.

A Derivations and Proofs

A.1 Wages To derive the wage rate under Nash bargaining, consider the household's problem:

$$J_t = \max_{c_t} c_t^{1-\gamma}/(1-\gamma) + \beta E_t J_{t+1}$$

subject to

$$c_t = w_t n_t + b u_t - \tau_t,$$
$$n_t = (1-\bar{s})n_{t-1} + f_t u_{t-1},$$
$$u_t = u_{t-1} + \bar{s} n_{t-1} - f_t u_{t-1}.$$

Using the envelope theorem, the marginal values of employment and unemployment are given by

$$J^H_{n,t} = w_t + E_t[x_{t+1}((1-\bar{s})J^H_{n,t+1} + \bar{s}J^H_{u,t+1})],$$
$$J^H_{u,t} = b + E_t[x_{t+1}(f_{t+1}J^H_{n,t+1} + (1-f_{t+1})J^H_{u,t+1})].$$

Similarly, use the firm's problem to define the marginal value of employment to the firm,

$$J^F_{n,t} = a_t - w_t + (1-\bar{s})E_t[x_{t+1}J^F_{n,t+1}] = \tfrac{\kappa-\lambda_{v,t}}{q_t}.$$

Define the total surplus of a new match as $\Lambda_t = J^F_{n,t} + J^H_{n,t} - J^H_{u,t}$. The equilibrium wage maximizes $(J^H_{n,t} - J^H_{u,t})^\eta (J^F_{n,t})^{1-\eta}$. Optimality implies $J^H_{n,t} - J^H_{u,t} = \eta\Lambda_t$ and $J^F_{n,t} = (1-\eta)\Lambda_t$. Combining the optimality conditions with $J^H_{n,t}$, $J^H_{u,t}$, and $J^F_{n,t}$, and defining tightness as $\theta_t = v_t/u_{t-1}$, we obtain

$$w_t = \eta(a_t + \kappa E_t[x_{t+1}\theta_{t+1}]) + (1-\eta)b.$$

A.2 The Efficient Allocation To solve for the efficient allocation, we imagine that the frictional labor market is controlled by a central planner who posts vacancies on behalf of firms, so it internalizes the two externalities associated with vacancy creation. The central planner solves

$$W_t = \max_{c_t, n_t, v_t} c_t^{1-\gamma}/(1-\gamma) + \beta E_t W_{t+1}$$

subject to

$$c_t = a_t n_t - \kappa v_t + b(1-n_t) - \tau_t,$$
$$n_t = (1-\bar{s})n_{t-1} + \mathcal{M}(1-n_{t-1}, v_t),$$
$$v_t \geq 0,$$

which imposes $u_t = 1 - n_t$. The efficient allocation is characterized by (1), (8), (10), and

$$\tfrac{\kappa-\lambda_{v,t}}{\mathcal{M}_v(1-n_{t-1},v_t)} = a_t - b + E_t[x_{t+1}\tfrac{\kappa-\lambda_{v,t+1}}{\mathcal{M}_v(1-n_t,v_{t+1})}(1-\bar{s}-\mathcal{M}_u(1-n_t,v_{t+1}))], \tag{A.1}$$

$$n_t = (1-\bar{s})n_{t-1} + \mathcal{M}(1-n_{t-1},v_t). \tag{A.2}$$

A.3 PROOFS

Recall $\mathcal{M}(u_t^s, v_t)$ is strictly increasing, strictly concave, and twice differentiable in both arguments, and it exhibits constant returns to scale. We use the following standard results:

Lemma 1. $\mathcal{M}_{vv}(1,\theta_t)\theta_t = -\mathcal{M}_{uv}(1,\theta_t)$.

Lemma 2. *The elasticity of substitution has the equivalent representation*

$$\sigma(\theta_t) = \frac{\mathcal{M}_v(1,\theta_t)\mathcal{M}_u(1,\theta_t)}{\mathcal{M}_{vu}(1,\theta_t)\mathcal{M}(1,\theta_t)}.$$

Proposition 1 A constant returns to scale matching function, $\mathcal{M}(u_t^s, v_t)$, has linear approximation

$$\mathcal{M}(u_t^s,v_t) \approx \mathcal{M}(\bar{u}^s,\bar{v}) + \mathcal{M}_u(\bar{u}^s,\bar{v})(u_t^s - \bar{u}^s) + \mathcal{M}_v(\bar{u}^s,\bar{v})(v_t - \bar{v}),$$

where $(\bar{u}^s, \bar{v})$ is the point of approximation (e.g., a model's deterministic steady state). By constant returns to scale, Euler's theorem implies $\bar{m} \equiv \mathcal{M}(\bar{u}^s,\bar{v}) = \mathcal{M}_u(\bar{u}^s,\bar{v})\bar{u}^s + \mathcal{M}_v(\bar{u}^s,\bar{v})\bar{v}$. Combining these results and converting the steady-state partial derivatives into matching elasticities yields

$$\mathcal{M}(u_t^s,v_t) \approx (1-\bar{\epsilon})\tfrac{\bar{m}}{\bar{u}^s}u_t^s + \bar{\epsilon}\tfrac{\bar{m}}{\bar{v}}v_t, \tag{A.3}$$

where $\bar{\epsilon}$ is the matching elasticity evaluated at the approximation point. However, (A.3) is also the first-order approximation of a Cobb-Douglas matching function $\mathcal{M}(u_t^s,v_t) = \phi(u_t^s)^\alpha v_t^{1-\alpha}$ with $\alpha = 1-\bar{\epsilon}$. Thus, using the Cobb-Douglas specification is without loss of generality up to first order.

Proposition 2 Differentiating the matching elasticity function $\epsilon(\theta_t) = \frac{\mathcal{M}_v(1,\theta_t)\theta_t}{\mathcal{M}(1,\theta_t)}$ yields

$$\epsilon'(\theta_t) = \left(\frac{\mathcal{M}_{vv}(1,\theta_t)\theta_t}{\mathcal{M}_v(1,\theta_t)} + 1 - \epsilon(\theta_t)\right)\frac{\mathcal{M}_v(1,\theta_t)}{\mathcal{M}(1,\theta_t)}.$$

Use Lemma 1 and Lemma 2 to obtain

$$\epsilon'(\theta_t) = \left(-\frac{1}{\sigma(\theta_t)}\frac{\mathcal{M}_u(1,\theta_t)}{\mathcal{M}(1,\theta_t)} + 1 - \epsilon(\theta_t)\right)\frac{\mathcal{M}_v(1,\theta_t)}{\mathcal{M}(1,\theta_t)}.$$

Replace $\frac{\mathcal{M}_u(1,\theta_t)}{\mathcal{M}(1,\theta_t)} = 1 - \epsilon(\theta_t)$ and rearrange to obtain

$$\epsilon'(\theta_t) = \frac{\sigma(\theta_t)-1}{\sigma(\theta_t)}(1-\epsilon(\theta_t))\frac{\mathcal{M}_v(1,\theta_t)}{\mathcal{M}(1,\theta_t)}. \tag{A.4}$$

Hence the sign of $\epsilon'(\theta_t)$ has the same sign as $\sigma(\theta_t) - 1$.

Corollary 1 Combine Proposition 2 with the fact that $\sigma(\theta_t) = \sigma$ for all $\theta_t > 0$.

Proposition 3 After imposing Assumption 1, (7) simplifies to

$$\frac{\kappa - \lambda_{v,t}}{q_t} = a_t - b + \beta(1-\bar{s})E_t\left[\frac{\kappa - \lambda_{v,t+1}}{q_{t+1}}\right].$$

We can guess and verify a unique solution of the form $\frac{\kappa-\lambda_{v,t}}{q_t} = \delta_0 + \delta_1(a_t - \bar{a})$, where

$$\delta_0 = \frac{\bar{a} - b}{1 - \beta(1-\bar{s})}, \qquad \delta_1 = \frac{1}{1-\beta(1-\bar{s})\rho_a}.$$

If $\lambda_{v,t} > 0$ then $v_t = 0$. Since $m_t = v_t$ and $q_t = 1$ for v_t arbitrarily close to 0, we have $q_t = 1$ when $\lambda_{v,t} > 0$ by continuity. Therefore, if productivity is such that $\kappa/(\delta_0 + \delta_1(a_t - \bar{a})) \in [0,1)$, then $q(a_t) = \kappa/(\delta_0 + \delta_1(a_t - \bar{a}))$ and $\lambda_{v,t} = 0$. Otherwise, $q_t = 1$ and $\lambda_{v,t} = \kappa - \delta_0 - \delta_1(a_t - \bar{a})$.

Proposition 4 Differentiate $\mu_q(\theta) = \mathcal{M}(1,\theta)/\theta$ to obtain $\mu_q'(\theta) = -\frac{1-\epsilon(\theta)}{\theta}\frac{\mathcal{M}(1,\theta)}{\theta}$. Hence

$$\theta'(a_t) = -\frac{q'(a_t)}{1-\epsilon_t}\frac{\theta(a_t)^2}{\mathcal{M}(1,\theta(a_t))}.$$

Use $q'(a_t) = -q(a_t)^2\delta_1/\kappa$ and $q(a_t)\theta(a_t) = \mathcal{M}(1,\theta(a_t))$, to obtain

$$\theta'(a_t) = \frac{\delta_1}{\kappa}\frac{\mathcal{M}(1,\theta(a_t))}{1-\epsilon_t} > 0.$$

Differentiate and use (A.4) to obtain

$$\theta''(a_t) = \frac{\delta_1}{\kappa}\frac{2\sigma_t - 1}{\sigma_t}\frac{\mathcal{M}_v(1,\theta(a_t))\theta'(a_t)}{1-\epsilon_t}. \tag{A.5}$$

Hence the sign of $\theta''(a_t)$ has the same sign as $\sigma_t - 1/2$.

Proposition 5 Differentiate $f'(a_t) = \mathcal{M}_v(1,\theta(a_t))\theta'(a_t)$ to obtain

$$f''(a_t) = \mathcal{M}_{vv}(1,\theta(a_t))\theta'(a_t)^2 + \mathcal{M}_v(1,\theta(a_t))\theta''(a_t).$$

Use Lemma 1 and Lemma 2 to obtain

$$f''(a_t) = \left(\theta''(a_t) - \frac{1}{\sigma(\theta)}\frac{\mathcal{M}_u(1,\theta)}{\mathcal{M}(1,\theta)}\frac{\theta'(a_t)^2}{\theta(a_t)}\right)\mathcal{M}_v(1,\theta(a_t)).$$

Replace $\frac{\mathcal{M}_u(1,\theta_t)}{\mathcal{M}(1,\theta_t)} = 1 - \epsilon(\theta_t)$ and use (A.5) to obtain

$$f''(a_t) = \left(\frac{\delta_1}{\kappa}\frac{2\sigma_t - 1}{\sigma_t}\frac{\mathcal{M}_v(1,\theta(a_t))}{1-\epsilon_t} - \frac{1-\epsilon_t}{\sigma_t}\frac{\theta'(a_t)}{\theta(a_t)}\right)\theta'(a_t)\mathcal{M}_v(1,\theta(a_t)).$$

Use $\theta'(a_t) = \frac{\delta_1}{\kappa}\frac{\mathcal{M}(1,\theta(a_t))}{1-\epsilon_t}$ and $\epsilon_t = \frac{\mathcal{M}_v(1,\theta_t)\theta_t}{\mathcal{M}(1,\theta_t)}$, to obtain

$$f''(a_t) = \frac{2\sigma_t\epsilon_t - 1}{\sigma_t}\frac{\mathcal{M}_v(1,\theta(a_t))(\theta'(a_t))^2}{\theta(a_t)}.$$

Hence the sign of $f''(a_t)$ is the same as the sign of $\sigma_t\epsilon_t - 1/2$.

Corollary 2 Recall that $q(a_t) \in (0,1)$. When the matching function is CES, we have $\sigma_t = \sigma$ and $\epsilon_t = (1-\vartheta)(\phi/q(a_t))^{(\sigma-1)/\sigma}$. By Proposition 5, the sign of $f''(a_t)$ depends on whether

$$\mathcal{F}_t \equiv 2\sigma(1-\vartheta)(\phi/q(a_t))^{(\sigma-1)/\sigma} \lesseqgtr 1.$$

Case 1 ($\sigma > 1$): $(\phi/q(a_t))^{(\sigma-1)/\sigma} \in \left(\phi^{(\sigma-1)/\sigma}, \infty\right)$, so $\mathcal{F}_t > 2\sigma(1-\vartheta)\phi^{(\sigma-1)/\sigma}$ for all feasible $q(a_t)$. Thus, $\sigma > \frac{1}{2(1-\vartheta)\phi^{(\sigma-1)/\sigma}} \geq 1$ implies $f''(a_t) > 0$ for all a_t such that $q(a_t) \in (0,1)$.

Case 2 ($\sigma < 1$): $(\phi/q(a_t))^{(\sigma-1)/\sigma} \in \left(0, \phi^{(\sigma-1)/\sigma}\right)$, so $\mathcal{F}_t < 2\sigma(1-\vartheta)\phi^{(\sigma-1)/\sigma}$ for all feasible $q(a_t)$. Thus, $\sigma < \frac{1}{2(1-\vartheta)\phi^{(\sigma-1)/\sigma}} \leq 1$ implies $f''(a_t) < 0$ for all a_t such that $q(a_t) \in (0,1)$.

Case 3 ($\sigma = 1$): $\sigma = 2(1-\vartheta) = 1$ implies $f''(a_t) = 0$ for all a_t such that $q(a_t) \in (0,1)$.

Corollary 3 Given the Den Haan et al. (2000) matching function, we have $\sigma_t = 1/(1+\iota)$ and $\epsilon_t = q(a_t)^\iota$. By Proposition 5, the sign of $f''(a_t)$ depends on whether

$$\mathcal{F}_t = 2q(a_t)^\iota/(1+\iota) \lesseqgtr 1.$$

Since $\iota > 0$, we have $2q(a_t)^\iota/(1+\iota) < 2/(1+\iota)$ for all feasible $q(a_t)$. Therefore $\iota > 1$ implies that $f''(a_t) < 0$ for all a_t such that $q(a_t) \in (0,1)$.

Proposition 6 Given wedges $\{\tau_{v,t}, \tau_{n,t}\}$, the firm's optimal vacancy creation condition becomes

$$\frac{\kappa - \lambda_{v,t}}{q_t} = \frac{1-\eta}{1+\tau_{v,t}}(a_t - b) + E_t\left[\tilde{x}_{t+1}\frac{\kappa - \lambda_{v,t+1}}{q_{t+1}}\left(1 - \bar{s} - \frac{1}{1+\tau_{v,t+1}}\frac{q_{t+1}}{\kappa - \lambda_{v,t+1}}(\kappa\eta\theta_{t+1} + \tau_{n,t+1})\right)\right],$$

where $\tilde{x}_{t+1} \equiv x_{t+1}(1+\tau_{v,t+1})/(1+\tau_{v,t})$. Setting

$$\tau_v(\theta_t) = (1-\eta)/\epsilon(\theta_t) - 1,$$
$$\tau_n(\theta_t) = \theta_t((\kappa - \lambda_{v,t})\tau_v(\theta_t) - \eta\lambda_{v,t}),$$

aligns the private optimality condition with the efficient condition (A.1). Differentiating yields

$$\tau_v'(\theta_t) = -(1-\eta)\epsilon'(\theta_t)/\epsilon(\theta_t)^2,$$
$$\tau_n'(\theta_t) = \kappa(\theta_t\tau_v'(\theta_t) + \tau_v(\theta_t)) = \kappa\left[\frac{1-\eta}{\epsilon(\theta_t)} - 1 - \frac{1-\eta}{\epsilon(\theta_t)^2}\theta_t\epsilon'(\theta_t)\right].$$

Since (A.4) implies $\epsilon'(\theta_t)\theta_t/\epsilon(\theta_t) = (\sigma_t - 1)(1 - \epsilon(\theta_t))/\sigma_t$, we obtain

$$\tau_v'(\theta_t) = -\left(\frac{1-\eta}{\theta_t}\right)\left(\frac{\sigma_t - 1}{\sigma_t}\right)\left(\frac{1-\epsilon(\theta_t)}{\epsilon(\theta_t)}\right),$$
$$\tau_n'(\theta_t) = \kappa\left[\frac{1-\eta}{\epsilon(\theta_t)} - 1 - (1-\eta)\left(\frac{\sigma_t - 1}{\sigma_t}\right)\left(\frac{1-\epsilon(\theta_t)}{\epsilon(\theta_t)}\right)\right].$$

Hence, $\tau_v'(\theta_t) > 0$ when $\sigma_t < 1$ and $\tau_n'(\theta_t) > 0$ when $\sigma_t < \frac{1-\eta}{\eta}\frac{1-\epsilon_t}{\epsilon_t}$ for all $\theta_t > 0$.

B SOLUTION METHOD

The equilibrium system of the model is summarized by $E[g(\mathbf{x}_{t+1}, \mathbf{x}_t, \varepsilon_{t+1})|\mathbf{z}_t, \mathcal{P}] = 0$, where g is a vector-valued function, $\mathbf{x}_t$ is a vector of variables, ε is a vector of productivity shocks, $\mathbf{z}_t$ is a vector of states, and $\mathcal{P}$ is a vector of parameters. There are many ways to discretize the productivity process. We use the Markov chain in Rouwenhorst (1995), which Kopecky and Suen (2010) show outperforms other methods for approximating autoregressive processes. The bounds on the state variable n_{t-1} are set to $[0.85, 0.98]$, which contains over 99% of the ergodic distribution. We discretize a_t and n_{t-1} into 7 and 21 evenly-spaced points, respectively. The product of the points in each dimension, D, is the total nodes in the state space ($D = 147$). The realization of $\mathbf{z}_t$ on node d is denoted $\mathbf{z}_t(d)$. The Rouwenhorst method provides integration weights, $\phi(m)$, for $m \in \{1, \ldots, M\}$.

Since vacancies $v_t \geq 0$, we introduce an auxiliary variable, μ_t, such that $v_t = \max\{0, \mu_t\}^2$ and $\lambda_{0,t} = \max\{0, -\mu_t\}^2$, where $\lambda_{v,t}$ is the Lagrange multiplier on the non-negativity constraint. If $\mu_t \geq 0$, then $v_t = \mu_t^2$ and $\lambda_{v,t} = 0$. When $\mu_t < 0$, the constraint is binding, $v_t = 0$, and $\lambda_{v,t} = \mu_t^2$. Therefore, the constraint on v_t is transformed into a pair of equalities (Garcia and Zangwill, 1981).

The following steps outline our nonlinear policy function iteration algorithm:

1. Use Sims's (2002) `gensys` algorithm to solve the linearized model. Then map the solution for the policy functions to the discretized state space. This provides an initial conjecture.
2. On iteration $j \in \{1, 2, \ldots\}$ and each node $d \in \{1, \ldots, D\}$, use Chris Sims's `csolve` to find $\mu_t(d)$ to satisfy $E[g(\cdot)|\mathbf{z}_t(d), \mathcal{P}] \approx 0$. Guess $\mu_t(d) = \mu_{j-1}(d)$. Then apply the following:
   (a) Solve for all variables dated at time t, given $\mu_t(d)$ and $\mathbf{z}_t(d)$.
   (b) Linearly interpolate the policy function, μ_{j-1}, at the updated state variables, $\mathbf{z}_{t+1}(m)$, to obtain $\mu_{t+1}(m)$ on every integration node, $m \in \{1, \ldots, M\}$.
   (c) Given $\{\mu_{t+1}(m)\}_{m=1}^{M}$, solve for the other elements of $\mathbf{x}_{t+1}(m)$ and compute

$$E[g(\mathbf{x}_{t+1}, \mathbf{x}_t(d), \varepsilon_{t+1})|\mathbf{z}_t(d), \mathcal{P}] \approx \textstyle\sum_{m=1}^{M} \phi(m) g(\mathbf{x}_{t+1}(m), \mathbf{x}_t(d), \varepsilon_{t+1}(m)).$$

   Set $\mu_j(d) = \mu_t(d)$ when `csolve` converges.

3. Repeat step 2 until $\mathrm{maxdist}_j < 10^{-7}$, where $\mathrm{maxdist}_j \equiv \max\{|\mu_j - \mu_{j-1}|\}$. When that criterion is satisfied, the algorithm has converged to an approximate nonlinear solution.

The algorithm is programmed in Fortran with Open MPI and run on the BigTex supercomputer.

C HOME PRODUCTION

In the baseline model, we set b to target the standard deviation of unemployment in our sample. This section shows we can equivalently set b externally as an unemployment benefit, and instead use home production to target unemployment volatility by following Petrosky-Nadeau et al. (2018).

The household derives utility from the consumption of the final market good $c_{m,t}$ and home production $c_{h,t}$. It has log utility over composite consumption $c_t = (\omega c_{m,t}^e + (1-\omega)c_{h,t}^e)^{1/e}$, where $\omega \in (0,1)$ is the preference weight on the final market good and $e \leq 1$ governs the elasticity of substitution $1/(1-e)$. The home production technology is $c_{h,t} = a_h u_t$, where $a_h > 0$ is productivity.

Household optimization yields the pricing kernel $x_{t+1} = \beta(c_{m,t}/c_{m,t+1})^{1-e}(c_t/c_{t+1})^e$. The flow value of unemployment becomes $z_t = a_h((1-\omega)/\omega)(c_{m,t}/c_{h,t})^{1-e} + b$, so the Nash wage satisfies

$$w_t = \eta((1-\alpha)y_t/n_t + \kappa(1-\chi\bar{s})E_t[x_{t+1}\theta_{t+1}]) + (1-\eta)z_t.$$

The other equilibrium conditions are unchanged from the baseline model described in Section 3.

We set $b = 0.4$ to reflect the value of unemployment benefits (Shimer, 2005), and set $a_h = 1$ to steady-state labor productivity in final good production. We then set $e = 1$, in line with existing calibrations and estimates (Benhabib et al., 1991; Petrosky-Nadeau et al., 2018). In this case, $z_t = (1-\omega)/\omega + b$, so ω determines the level of z, and hence the volatility of unemployment following the fundamental surplus arguments in Ljungqvist and Sargent (2017). Thus, we can set ω in each model to generate the same unemployment volatility and quantitative results as the baseline model.